Schumpeterian business cycles

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Abstract

This paper presents an economy where business cycles and long term growth are both endogenously generated by the same type of iid shocks. I embed a multi-sector real business cycle model into an endogenous growth framework where innovating firms replace incumbent production firms. The only source of uncertainty is the imperfectly observed quality of innovation projects. As long as the goods are complements, a successful innovation in one sector increases demand for the output of other sectors. Higher profits motivate higher innovation efforts in the other sectors. The increase in productivity in one sector is thus followed by increases in productivity in the other sectors and the initial innovation generates persistent movement in aggregate productivity.

1 Introduction

One of the most prominent characteristics of aggregate macroeconomic time-series like output is the high degree of persistence. However, the overwhelming majority of macroeconomic models do not generate such persistence endogenously. Instead, as observed by Cogley and Nason (1995), a wide range of models -from small calibrated

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models such as Kydland and Prescott (1982) to large scale estimated models, such as those recently studied by Christiano, Motto and Rostagno (2014)- require a highly auto-correlated exogenous productivity process to match the data. If the persistence in the exogenous driving force is crucial for our models, where does it come from?

To address this question, I propose a mechanism that transforms white noise shocks affecting endogenous R&D activity into persistent business cycle fluctuations. To assess this mechanism quantitatively, I build a multi-sector business cycle model with endogenous growth and calibrate it to match traditional business cycle statistics, as well as empirical growth patterns.

To evaluate the strength of the internal propagation mechanism in generating the persistence, I simulate the model and compare the moments with the targets in the data and a benchmark RBC model which is driven by a productivity process with 0.95 autocorrelation. Despite having no persistence in the shock process, the model generates autocorrelation in output of 0.6 (compared to 0.7 generated by the RBC model). Furthermore, the spectrum shows that most of the volatility is generated at business cycles frequencies.

In the model, economic growth is the aggregate result of the accumulation of successful innovations at the micro-level over time. The innovation is accompanied by an incumbent production firm being replaced by a newcomer (whom I call *a research firm*) who made a risky investment to develop a better production capacity that turned out to be successful. This is a classical Schumpeterian creative destruction mechanism. The innovation itself is modelled as the capacity to produce the same good with lower marginal costs, where both the probability of success (gaining access), as well as the size of innovation (the difference in marginal costs between the new entrant and the incumbent) depend on the quality of the project and the optimally chosen research effort. The only source of uncertainty is the imperfectly observed quality of the research projects available to research firms, which are modelled as independent across time and firms.

The persistence on the aggregate level comes from the relationship between the optimal research effort and the dispersion of relative productivity across sectors. Assuming that the consumer views goods from different sectors as complements, successful innovation in one sector increases the demand for goods of other sectors. This creates a positive externality, since higher demand translates into higher profits; it therefore incentivises research in the less productive sectors. While research increases (decreases) in the least (the most) productive sectors, in a simple two-period model, it can be shown analytically that the combined effect is positive, i.e. the increased dispersion in productivity leads to more research overall. I show that this is still the case in the full general equilibrium model with more sectors, which is solved numerically. The persistence is then the outcome of changes in the expected growth; if starting from a symmetric situation one sector becomes lucky in innovation and becomes more productive; the increase in productivity dispersion stimulates total research which increases the expected productivity growth until the gap in relative productivity is closed.

I assume that the innovation process is labour-intensive and that the innovation outcome is additive in the unobserved quality of the project, and a concave function of the labour input. In this framework, the signal about the quality of the research project can be viewed as a *news shock* in the business cycle literature because it is informative about future productivity. In the standard news shock literature (Barsky and Sims, 2011; Schmitt-Grohe and Uribe, 2012), a news shock does not directly affect production on impact. This is why labour typically falls in response to good news (because of the wealth effect), which is regarded as counter-intuitive (Beaudry and Portier, 2004). In the present model, however, good news effectively shift expected returns to "production" of research, and labour hence increases. Furthermore, the present paper is also explicit about how the news of the future gets revealed and thus potentially provides a wider set of testable predictions.

The present model poses a challenge for standard solution methods. First, the innovation is implemented only if the research outcome is *sufficient* for the research firm to replace the incumbent production firm. This introduces a kink into the optimisation problem of both types of firms as well as that of the households. Second, the three problems (household's, production and research firms') are too large to be solved simultaneously, so the solution is obtained iteratively. Finally, the existence of a finite number of sectors makes the computation of expectations non-trivial, particularly in combination with kinks. To address these complications, I develop a framework which allows for a global solution of the model using a projection algorithm. In particular, I extend existing procedures dealing with highly dimensional state-spaces (Judd, Maliar and Maliar, 2012) by developing a method which adaptively extends and relocates the grid used in the projection. Moreover, I propose a novel method that computes expectations of functions with kinks in such multi-dimensional settings. To the best of my knowledge, this is the first paper which solves a multi-sector business cycle model with endogenous growth using a global method.

One important quality of this numerical framework is its ability to solve the model allowing for *an endogenously* determined *stochastic* growth rate. The former feature enables studying the interaction between shorter run business cycle fluctuations and long term growth and the latter allows the long term growth to be an outcome of innovations, which are individually unlikely, but accumulate over sectors and time. This is particularly important for a model of endogenous growth; suppose that there is a finite number of innovators who succeed only for a particularly high realisation of some shock. In such a setting there would be a successful innovation occasionally and hence this economy would be growing. However, deterministic steady state is misleading because switching off the shock would switch off the growth as well.

The present paper is close in spirit to Comin and Gertler (2006) who build a model where R&D amplifies shocks and show how it can contribute to cycles over traditional business as well as lower frequencies. In Comin and Gertler's model, the R&D and innovation adoption plays only the role of amplification of other shocks, whereas in the present model, the focus is on the stochastic outcomes and innovation plays the role of both the driving shock and amplification. Furthermore, they still use a persistent exogenous driving process to match the data, whereas the aim of the present paper is to show that the non-persistent innovation shocks *alone* can lead to persistent aggregate fluctuations.[1]

If changes in productivity are the results of innovation, then variations in the resources devoted to R&D should be able to help explain cyclical fluctuations. While this relationship is difficult to measure due to noisiness of R&D data, there is mild consensus that R&D is pro-cyclical (Barlevy, 2007; Walde and Woitek, 2004). On a disaggregated level, empirical studies focus on either firm or industry variables.[2] R&D spending can be used as an instrument for actual innovation. To solve the lags between discovery and implementation of a particular innovation, Alexopoulos (2011) constructs an index of technological innovation using the publication of manuals as a measure of innovation. These publications are expected to appear exactly around the time when an innovation is being introduced into production. Alexopoulos finds a positive correlation between this index and TFP, and between capital investment and labour.

Among other papers studying business cycles fluctuations in the endogenous growth framework, the most similar is the setting of Phillips and Wrase (2006).[3]

[1]The shock Comin and Gertler (2006) use is a AR(1) process of wage markups with the autoregressive coefficient of 0.6. The fact that this number is lower than standard 0.95 for productivity process shows that the amplification mechanism in their model is stronger.

[2]See Griliches (1998) for an overview.

[3]For other related models, see Jones, Manuelli, Siu and Stacchetti (2005b); Jones, Manuelli and Siu (2005a); Maliar and Maliar (2004); Ozlu (1996); Lambson and Phillips (2007); Andolfatto and MacDonald (1998); Wälde (2005).

Phillips and Wrase also build a model with a finite number of sectors with endogenous innovation, but there are important differences in the setting which lead to different outcomes. In particular, the complementarity of goods, which plays a crucial role in my model, is absent. This means that the relative productivity channel cannot operate, which diminishes the possibility for a idiosyncratic shocks to play an aggregate role.

Given the prominent role that complementarity plays in the model, it is worth examining it in detail. In reality, there are varying degrees of complementarity for different goods. The higher degree of aggregation, the higher the degree of complementarity there is. For example, studies looking at the tradable versus non-tradable sectors (Stockman and Tesar, 1995), the capital-intensive versus non-intensive sectors (Acemoglu and Guerrieri, 2008), and services-manufacturing-agriculture (Ngai and Pissarides, 2008; Herrendorf *et al.*, 2013) find the elasticity of substitution below one.[4] At the same time, at a much lower degree of aggregation, the goods within finer product categories are undoubtedly substitutes. The Schumpeterian framework presented in this paper is an attempt to capture both. On one hand, the goods of different sectors are viewed as complements to capture the complementarity between products of very different types. On the other hand, within one sector, the creative destruction, whereby one producer is replaced by another, can be interpreted as two producers producing different but perfectly substitutable goods.

Finally, there is literature studying the aggregate effects of idiosyncratic shocks. The law of large numbers implies that as the number of sectors increase, the aggregate fluctuations shrink (Dupor, 1999). However, Horvath (1998) shows that if some sectors are more important in the input-output structure of the economy, then the shocks to these sectors do not average out. Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012) formalize this notion and derive the rate of decay based on the network structure of the input-output relationships among different sectors. Further, Holly and Petrella (2012) provide empirical evidence of such network effects based on a sample of US manufacturing firms. In these models, it is the *supply* disruption of the systemically important sectors which gets amplified to the aggregate level. In my model there is only a finite number of sectors, but also the importance of shocks is generated via *demand*: an innovation in the least productive sector affects the aggregate variables much more than an innovation of equal size in the most productive sector because of the complementarity.

The structure of the paper is the following. I first introduce the model and present

[4] More detailed overview of this literature in provided in the calibration section.

the mechanism analytically in a simplified setting. Then, I describe the full model and my solution procedure. I calibrate the model and present the results. Finally, I conclude. The appendix contains details on the derivation of model equation and additional results.

2 Model intuition

To provide intuition for the mechanism in the model, let's consider a simple example of a Robinson Crusoe economy. Out of all food a tropical island has to offer, Robinson Crusoe enjoys fish curry the most. In order to prepare it, he needs to catch some fish and harvest coconuts. Robinson has some knowledge about either activity but he also knows that he could try to search for a better place to fish or new coconut trees or both. Looking for either can potentially increase his productivity in the respective activity. Although the outcome of either search is uncertain, Robinson knows that the more effort he puts in, the bigger the improvement odds are.

To gain some intuition for the business cycle model, I first study Robinson's problem in a two period setting. In particular, it can be shown that there is a negative relationship between the relative productivity of a given activity and the optimal research effort trying to increase respective productivity. Under certain conditions, I am then able to show that there is a positive relation between the total effort in research and the dispersion of productivities.

Three factors play an important role for the optimal research effort. First, how effective is Mr. Crusoe already in obtaining either resource? Second, what is the degree of complementarity between fish and coconut? The stronger is the complementarity, the more balanced the consumption bundle should be. Third, what is the likelihood of successful innovation in either activity and what benefit would such an innovation bring? Let's write down the model formally to shed some light on the problem.

In the two period setting, Robinson starts with some productivity which is fixed in the first period, but the research effort might affect his productivity in the second period. To formally define the problem, let the utility be $u(c_1, c_2, l_1^f, l_2^f, l_1^n, l_2^n, r_1^f, r_1^n)$, where the subscripts denote the period, c represent the curry consumption, which is a bundle of coconuts n and fish f. l^n and l^f denote the labour costs of harvesting coconuts and fish and finally, r^n and r^f represent the time spent improving the productivity in obtaining the curry ingredients. I assume that the optimal proportion in the curry recipe can be approximated by constant elasticity of substitution preferences. The production function for fish and coconuts is assumed to be labour intensive with decreasing returns to scale.

In this setting, research increases the productivities in the second period, a_2^n and a_2^f, relative to the values fixed in the first period. A successful innovation increases either productivity by a fixed factor $\bar{e}$, so that if the innovation in fishing is successful, $a_2^f = (1+\bar{e})a_1^f$. Assume that Robinson has an idea every morning how to improve the productivity of either activity and that he also has a vague idea of how likely it is that this innovation will work μ. This μ represents how good the innovation project is and it will be the only shock driving the model. I further assume that the probability of innovation success is $\Phi(\mu^i + r^i)$, where $i \in \{f, n\}$ and Φ is the normal cumulative distribution function. In this setting, the higher likelihood of innovation comes either from having a better idea to start with (higher μ^i, which cannot be affected by Robinson), or doing more research (increasing r^i).

This two period problem can be written as

$$
\begin{aligned}
\max_{l_1^f, l_2^f, l_1^n, l_2^n, r_1^f, r_1^n} \quad & \mathsf{E}u(c_1, c_2, l_1^f, l_2^f, l_1^n, l_2^n, r_1^f, r_1^n), \\
\text{such that} \quad & \mu^f, \mu^n, a_1^f, a_1^n, \bar{e} \text{ given} \\
& f_t = a_t^f (l^f)^\alpha, \quad n_t = a_t^n (l^n)^\alpha, \\
& c_t = \left[\frac{f_t^{\frac{\theta-1}{\theta}} + n_t^{\frac{\theta-1}{\theta}}}{2} \right]^{\frac{\theta}{\theta-1}}, \\
\mathsf{P}\left[a_2^f = (1+\bar{e})a_1^f\right] = \Phi(\mu^f + r^f), & \quad \mathsf{P}\left[a_2^f = a_1^f\right] = 1 - \Phi(\mu^f + r^f), \\
\mathsf{P}\left[a_2^n = (1+\bar{e})a_1^n\right] = \Phi(\mu^n + r^n), & \quad \mathsf{P}\left[a_2^n = a_1^n\right] = 1 - \Phi(\mu^n + r^n).
\end{aligned}
$$

The expectation is taken with respect to the uncertain outcome of the innovation. In this two period two goods setting some results can be shown analytically.[5] First, it is optimal to do more research in the relatively less productive activity as it makes the resulting bundle more even and creates higher utility for complementary goods. More importantly, Robinson is going to spend more time researching in total the bigger the relative difference is between his productivity in catching fish and harvesting coconuts; in other words it pays off to close the gap in relative productivity without reducing the research of the more productive activity by too much.

Result 1. *The unconditional aggregate research is an increasing function of the*

[5]These results can be obtained without actually finding the values for the optimal research effort r^n and r^f.

dispersion of relative productivity. (For proof see the appendix D.2, page 54)

This means that the bigger is the difference between the productivities in the two activities, the more it pays off to do research. The reason is that the CES consumer values a balanced consumption bundle. This leads to a fall in research in the relatively more productive sector and increase in the relatively less productive sector. However, the result establishes that the total effect on research is positive.

In the full general equilibrium model, the value of different goods is communicated via the price mechanism; the goods in lower supply will command a higher price and such firms can earn higher profits. The research firms will adjust their efforts accordingly, which leads to more research being done in relatively less productive sectors.

The persistence in changes in productivity on the aggregate level is generated by the following mechanism. Starting from a symmetric situation where all industries are equally productive, a successful innovation makes one sector more productive than the others. Higher dispersion in productivity changes the incentives to conduct research in different sectors; the returns to successful innovation fall in the relatively more productive sectors and vice versa. However, the fall in research in the most productive sectors is more than compensated by the increase in the research in the least productive sectors, and therefore the aggregate research increases. With higher aggregate research, the growth rate is higher and the relatively less productive industries catch up. Therefore big jumps in productivity (which happen due to sampling variation) are followed by times with higher average growth. Hence the growth rate is persistent.

This means that one innovation shock generating a change in productivity leads to a prolonged period where there is higher chance of more changes in productivity. Change in productivity is hence persistent. Persistent changes in productivity then generate persistent changes in output and other endogenous variables.

The full model presented in the next section generalises this setting into a decentralised, infinite horizon, multi-sector model with capital and general balanced-growth path consistent utility function, where the innovation step e is not fixed.

3 Model

The basic structure of the model is depicted in figure 1. There are three types of agents. There is one representative household and there are N industries, each with one production firm and one research firm. The production firms produce

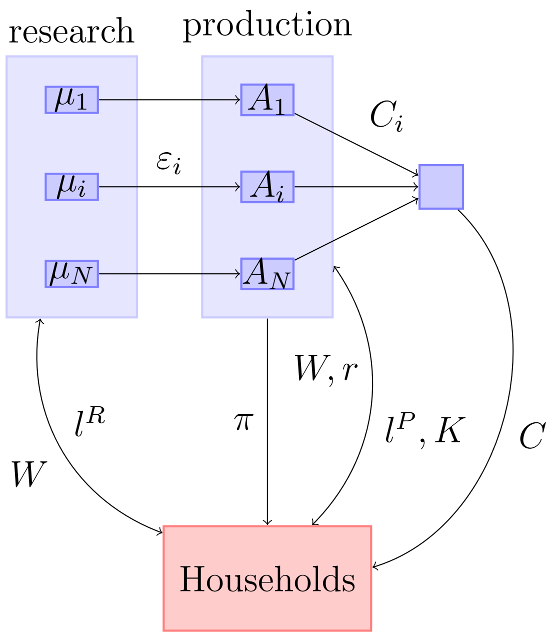


Figure 1: Basic scheme of the model

differentiated output. The research firms are trying to innovate and, if successful, will replace the production firm in their industry. Each agent type has different life-span: the household lives forever, a production firm lives until it is replaced by a successfully innovating research firm and finally, every period a new research firm is born in every industry and is either successful in innovation (and it replaces the corresponding production firm starting the next period), or it is unsuccessful and leaves.

Future productivity is determined by the innovation success of the research firms. Using labour, these research firms produce some innovation, which can be either successful or not. Importantly, in each period each research firm receives an industry specific signal about the quality of its research project. These signals play the role of news shocks, because they are informative about the expected future productivity. However, unlike standard news shocks, they directly affect the "production" function of the research firms *today*. In particular, research firms decide about optimal labour inputs and a better signal makes a research firm hire more researchers. Via this the mechanism, the model overcomes the problem of Pigou cycles. In standard RBC models, news about the future higher productivity decrease labour supply today due to a positive wealth effect, whereas here, the increase of labour demand in the research sector outweighs the wealth effect.

Before starting with the formal introduction of the model, let me start with one notation convention. The successful innovation increases productivity which leads to growth of some of the variables in the model. In what follows, I will denote such trending variables by upper case letters to contrast them against variables which do not grow over time, which are denoted by lowercase letters. For example, C_t is the level of consumption, where as l_t^P is the labour working in production firms in period t. In section 3.5, I show that there is an aggregate measure of productivity $\tilde{A}$, such that $x_t \equiv X_t/\tilde{A}_t$ is stationary for all upper case variables X. The lower case labeled variables are therefore stationary, either because they were stationary from the start (like l_t^P), or because they represent a normalised variable (for example $c_t \equiv C_t/\tilde{A}_t$).

I start the exposition with the problem of households, followed by the production firm description. These two are standard and it is the problem of the research firm which is crucial for the mechanism at the heart of the model, but this ordering allows for natural build up of the notation. All variables are in real terms.[6]

3.1 Household

The households work, rent out capital, consume, accumulate capital and invest in research and production firms.

3.1.1 Utility function

The representative household has preferences over consumption C_t and labour l_t

The household's problem is to maximize its discounted expected utility

$$\max_{\{C_t, l_t^P, l_t^R\}_{t=0}^{\infty}} \mathsf{E}\left[\sum_{t=0}^{\infty} \beta^t u(C_t, l_t^P, l_t^R)\right],$$

where $l_t^P = \frac{1}{N}\sum_{i=1}^{N} l_{it}^P$ is the labour supply to the production firms and $l_t^R = \frac{1}{N}\sum_{i=1}^{N} l_{it}^R$ is the labour supply to the research firms. The $1/N$ normalisation keeps the size of the household constant relative to the number of industries.

The instantaneous period utility function $u(C, l_t^P, l_t^R)$ of the household is assumed to be

$$u(C, l_t^P, l_t^R) = \frac{C_t^{1-\gamma}}{1-\gamma}(1-(l_t^P + l_t^R))^{-\phi}, \quad \gamma > 1, \phi > 0, \tag{1}$$

where $l_t^P, l_t^R > 0$ and $l_t^P + l_t^R < 1$. This implies that the research labour force is fully substitutable by the production labour force. Consumption C_t corresponds to a CES

[6] For more details, see appendix A.

consumption aggregate

$$C_t = \left[\frac{1}{N} \sum_{i=1}^{N} C_{it}^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}}, \tag{2}$$

where N is the number of industries/products. The weighting makes sure that a mere increase in the number of industries does not automatically increase utility and eliminates love for variety. In this paper, the number of industries is exogenous, so this normalisation has no economic implications. The aggregator implies that the optimal demand for consumption good i, C_i is a fraction of aggregate consumption C given by the relative price $P_i^{-\theta}$:

$$C_{it} = P_{it}^{-\theta} C_t. \tag{3}$$

The details of the derivation are given in appendix A.1, page 47.

3.1.2 Household budget constraint

The household earns wages by working in the production and research firms $W_t \sum_{i=1}^{N} (l_{it}^{P} + l_{it}^{R})$, capital income $r_t \sum_{i=1}^{N} K_{it}$ earned by renting out the capital and receives dividends from owning shares of the production firms (the fraction denoted by s_{it}^{P}). The production firms distribute all their profits, so households obtain $\sum_{i=1}^{N} \Pi_{it} s_{it}^{P}$. Furthermore, the household can potentially sell the shares of the production firms at price Q_{it}^{P} and the accumulated capital $(1-\delta)K_t$.

On the expenditures side, the household buys consumption C_t, accumulates capital K_{t+1} and buys shares s_{it+1}^{R} and s_{it+1}^{P} of research and production firms at prices Q_{it}^{R} and Q_{it}^{P}. One unit of capital K_t can costlessly be transformed into one unit of consumption good C_t.

When born, research firms issue equity to finance the research labour. Although only production firms generate profits (which are paid out as dividends), ownership of a research firm is also valuable. The reason is that the research firm has a chance to become a production firm, should the innovation be successful. I assume that the research firms are operated by a mutual fund, so that the research objective is maximize its value.[7] When distributing profits, production firms pay s^P fraction

[7] Because the mutual fund is ultimately also ultimately owned by the households, this separation prevents the research firms to act strategically in order to achieve other objectives the household might care about (employment, etc...). To model such strategic behavior or moral issues connected with the financing would be beyond the scope of this paper.

directly to the households. The rest of the profits, denoted by D, is paid out to the mutual fund, which then distributes the money back to the households. This means that the household receives all the profits every period, while the value of assets can be determined in a traditional way.

The budget constraint is (for derivation see appendix A.2 on page 48)

$$
\begin{aligned}
C_t + K_{t+1} - (1-\delta)K_t = r_t K_t + W_t(l_t^P + l_t^R) + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P \left(\Pi_{it} + Q_{it}^P\right) \\
- \frac{1}{N}\sum_{i=1}^{N} \left(s_{it+1}^P Q_{it}^P + s_{it+1}^R Q_{it}^R\right) + D_t,
\end{aligned}
\tag{4}
$$

where the household chooses C_t, K_{t+1} and s_{it+1}^P, s_{it+1}^R, taking W_t, r_t, Q_{it}^P and Q_{it}^R as given (for all industries $\forall i = 1, \ldots, N$). Success of a research firm has two implications. First, the incumbent firm is replaced and hence its stocks lose all value. Second, s_{it+1}^R becomes s_{it+1}^P, because the research firm now becomes the production firm and the owners do not change.[8]

Using the lagrangian, the optimality conditions (for detailed derivation see see appendix A.3 on page 48) for the labour supply are the following:

$$
-u_{C_t} = \frac{u_{l_t^P}}{W_t}.
$$

Let $m_{t+1} = \beta \frac{u_{C_{t+1}}}{u_{C_t}}$ be the stochastic discount factor, then the Euler equation can be written as

$$
1 = \mathsf{E}\left[m_{t+1}\left(r_{t+1} + 1 - \delta\right)\right],
$$

giving the condition on expected return to saving in the capital stock. This return has to be equal to the alternative saving sources, investing into stocks of research and production firms. For each industry $i = 1, \ldots, N$, the expected returns must satisfy

$$
1 = \mathsf{E}\mathbb{1}_{it} m_{t+1} \frac{\Pi_{it+1} + Q_{it+1}^P}{Q_{it}^R}, \tag{5}
$$

$$
1 = \mathsf{E}\bar{\mathbb{1}}_{it} m_{t+1} \frac{\Pi_{it+1} + Q_{it+1}^P}{Q_{it}^P}. \tag{6}
$$

[8] It is convenient to define an indicator $\mathbb{1}_{it}$ capturing the success of innovating research firm in sector i and conversely, the complementing indicator $\bar{\mathbb{1}}_{it} \equiv 1 - \mathbb{1}_{it}$ to capture the situation where the research firm is not successful and hence the production firm survives. Formal definitions of these indicator functions is given in equations (24) and (22).

Using the convention $l_t^P = \frac{\sum_i l_{it}^P}{N}$ and $l_t^R = \frac{\sum_i l_{it}^R}{N}$, the labour supply condition is

$$C_t = \frac{\gamma - 1}{\phi} \left(1 - \left(l^P + l^R\right)\right) W_t.$$

The stochastic discount factor is

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{1 - (l_{t+1}^P + l_{t+1}^R)}{1 - (l_t^P + l_t^R)}\right)^{-\phi}.$$

3.2 Production firm

Successful innovation leads to two firms with different marginal costs trying to maximize their profits in a sector. It turns out that if the complementarity is strong enough, the optimal strategy of the more productive firm is to price a limit price (a price equal to the marginal costs of the incumbent) and serve the whole market. To show that this is the case, I first derive the marginal cost function of a production firm. The cost function will be used in the next section as one of the inputs into the decision of the research firm. The full description of the optimal behavior of a production firm (i.e. what quantity it chooses) will be possible only after describing the innovation process and resulting mark-ups.

Suppose that the (real) output the production firm wants to produce is Y_{it}. Then the cost minimisation problem is simple; given the (real) factor prices (wage rate W and interest rate r) it chooses labour and capital to maximize its profits, subject to a limit pricing constraint to drive less productive competitors out of the market. The problem is simple, because the optimisation is static (the capital is accumulated by the household and only rented out to production firms).[9] The production function in each industry is standard Cobb-Douglas

$$Y_{it} = K_{it}^{\alpha} \left(A_{it} l_{it}^P\right)^{1-\alpha},$$

where A_{it} is the productivity of the production firm in industry i in period t. The problem is to find

$$\min_{l_{it}^P, K_{it}} \quad W_t l_{it}^P + r_t K_{it},$$
$$\text{s.t. } K_{it}^{\alpha} \left(A_{it} l_{it}^P\right)^{1-\alpha} = Y_{it}.$$

[9] Therefore if a production firm is replaced and exits the market, the capital stock is not affected.

where W and r denote real wage rate and interest rate. The problem is identical in all industries (apart from differences in A_{it}).

I assume away a strategic behavior affecting the decisions of individual firms, so the firms still take aggregate prices as given when they decide about their optimal price. This allows to solve the model easily and the higher is the number of industries, the less problematic this assumption is.

Given the fact that different production firms in different industries share the same Cobb-Douglas production function (they differ in their productivity A_{it}), the ratio of factors is the same for all of them,

$$\frac{K_{it}}{l_{it}^P} = \frac{W_t}{r_t}\frac{\alpha}{1-\alpha}, \tag{7}$$

so higher W relative to the r increases K relative to l^P. It is important to note that the optimal capital/labour ratio is always the same for all production firms, $K/l^P = \frac{W}{r}\frac{\alpha}{1-\alpha}$, no matter the firm specific productivity A_{it} is.

Next, this result is substituted into the production function to get $Y(l^P)$, which can be inverted to get $l^P(y)$ and $K(y)$, taking W, r as given. The cost function is $\psi(Y) = Wl^P(Y) + rK(Y)$, the marginal costs are then $\mathrm{d}\psi/\mathrm{d}Y = W\mathrm{d}l^P/\mathrm{d}Y + r\mathrm{d}K/\mathrm{d}Y$. Solving this yields

$$MC(Y_{it}) = A_{it}^{\alpha-1}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}. \tag{8}$$

The marginal costs are scale invariant (with respect to Y), so different firms (in different industries) differ only due to the difference in the productivity A_{it} regardless of their level of production Y_{it}. As expected, marginal costs are rising in W and r and decreasing in firm productivity,

$$l^P(Y_{it}) = Y_{it}\left(\frac{W_t}{r_t}\frac{\alpha}{1-\alpha}\right)^{-\alpha} A_{it}^{-1+\alpha}, \tag{9}$$

$$K(Y_{it}) = Y_{it}\left(\frac{W_t}{r_t}\frac{\alpha}{1-\alpha}\right)^{1-\alpha} A_{it}^{\alpha-1}. \tag{10}$$

3.3 Research firm

In my model, each period a new research firm is born in every production sector and tries to innovate upon the existing production technology. If the research firm is successful in innovation, it becomes a production firm in its industry in the following period. I will show that if the complementarity is strong enough, then it is optimal to charge the limit price and thereby drive the incumbent production firm out of the market (see result 2). It will therefore be the only production firm in its industry and will gain profits until it is replaced by another successful research firm in the future.

3.3.1 Research production function

To simplify the problem, let's transform the setting in the following way. Instead of letting the research firm observe *an informative but noisy signal* about the quality of a research project, it is more convenient to assume that the quality of the project has two parts, one that is completely observed by the research firm, denoted by μ and one that is completely unobserved, denoted by ε. One possible interpretation of the unobserved shock ε is the *luck* component of any innovative activity. The noisy signal setting can be replicated by μ and ε shocks if the mean and variance of ε is a function of the (observed) realisation of μ. I assume that μ and ε are independent, both within one sector and across sectors and time.

The success of a research firm is determined by three factors: *first*, by the observed quality of the project μ, *second*, by the amount of labour employed to improve the project l^R and *finally* an idiosyncratic shock ε that represents luck. The timing is crucial: ε is observed/realised only after l^R has been chosen.

The final research output is then $\mu + f\left(l^R\right) + \varepsilon$, where $f(\cdot)$ is the research production function. An innovation project is successful if this research output is positive. If it is negative the innovation failed and the research firm exits the economy. Innovation steps are defined as

$$e_{it+1} = \max\left\{0, \mu_{it} + f\left(l_{it}^R\right) + \varepsilon_{it}\right\}. \tag{11}$$

If the innovation is successful, $e_{it+1} > 0$, the research firm enters the production sector in the next period with a productivity bigger than that of the current production firm by the factor $1 + e_{it+1}$.

Let the production function for research be

$$f\left(l_{it}^R\right) = \kappa\left(l_{it}^R\right)^{\iota}, \tag{12}$$

so that the research production function has diminishing returns to scale parametrised by ι with slope coefficient κ. This is in line with literature which typically assumes that innovations are generated using labour only.

I assume that both shocks are distributed normally, $\mu_{it} \sim N(\mu_0, \sigma_\mu^2)$ and $\varepsilon_{it} \sim$

$N(0, \sigma_\varepsilon^2)$. μ_0 is parametrised to be a negative number. This means that without any labour effort, the chances of innovation are low. Formally, the probability of the innovation being successful can be found by finding the threshold value $\bar{\varepsilon}$, such that $\forall \varepsilon_{it} > \bar{\varepsilon}$, the research satisfies $f\left(l_{it}^R\right) + \mu_{it} + \varepsilon_{it} > 0$. Clearly $\bar{\varepsilon} = -\left(f\left(l_{it}^R\right) + \mu_{it}\right)$ and using the properties of normal distribution the probability of innovation success can be found to be

$$\mathsf{P}(e_{it+1} > 0 | \mu_{it}) = \Phi\left(\frac{\mu_{it} + f\left(l_{it}^R\right)}{\sigma_\varepsilon}\right), \tag{13}$$

where Φ is the standard normal cumulative distribution function. Similarly, the unconditional expected innovation step can be found using standard results about truncated normal distributions. The unconditional innovation step will be closely related to the aggregate growth rate generated by my model and this fact will be used to calibrate $\mu_0, \sigma_\mu, \sigma_\varepsilon, \kappa$ and ι.

3.3.2 Pricing

A successful research firm enters the production sector in the following period and sets its price in *limit pricing*. This means that it sets the price so that it drives the incumbent production firm out of the market by charging a price equal to the incumbent's marginal costs of production.

Marginal costs of producing output volume y do not depend on y because of constant returns to scale in both input factors, see equation (8). The difference between the marginal costs of the incumbent and those of the successful research firm is that the latter is more productive by the factor $1 + e_{it+1}$, the recent enhancement in productivity. The productivity of the recently successful research firm A_{it} therefore has the form

$$A_{it} = (1 + e_{it})\tilde{A}_{it}, \tag{14}$$

where $\tilde{A}_{it}$ is the productivity of the incumbent production firm. The marginal cost of the successful research firm (MC) relative to those of the incumbent firm ($\widetilde{MC}$) hence satisfy

$$\widetilde{MC} = (1 + e)^{1-\alpha} MC.$$

Therefore, in order to drive out the incumbent firm, the new entrant can charge a mark-up of $(1+e)^{1-\alpha}$.

Note that the incumbent firm and the new entrant use the same ratio of K/l (see equation (7)). The incumbent firm just needs to employ *more of both* factors (in the same proportion) to produce the same amount of output.

Hence the price charged for goods in sector i is

$$P_{it} = \frac{1}{\tilde{A}_{it}^{1-\alpha}} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{r_t}{\alpha} \right)^{\alpha}. \tag{15}$$

This price assumes that the production firm uses the optimal choice of inputs. Note that a higher productivity of the incumbent firm $\tilde{A}$ forces the new firm to charge a lower price. Due to the limit pricing, it is *the previous, not the current*, generation of firms whose productivity determines prices.

Result 2. *Limit pricing is a optimal strategy as long as $\theta \in (0,1)$.*

Proof. First, I show that any price p^* above the limit price $\tilde{p}$ cannot be optimal. If $p^* > \tilde{p}$, then the incumbent is present in the market, produces a positive amount of goods and generates profits. However, the incumbent can also charge a price marginally lower, gain the whole market. Because the entrant is earning positive profits if he charges the limit price and zero profits if she deviates upward, such deviation cannot be an equilibrium behavior.

Second, let's investigate the situation for prices lower than the limit price. In such a situation, the entrant is serving the whole market. Suppose that the new entrant chooses y^* which corresponds to price p^*. However, with $\theta < 1$, a small decrease in y^* increases profits because the price increase more than compensate for the loss in quantity sold. Hence any $p^* < \tilde{p}$ cannot be equilibrium, *(for more details, see appendix D.1, page 53)*. □

Given the CES consumption aggregator, the standard price index has the form of

$$1 \equiv P_t = \left[\frac{1}{N} \sum_{i=1}^{N} P_{it}^{1-\theta} \right]^{\frac{1}{1-\theta}} = W_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{\tilde{A}_i^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}, \tag{16}$$

and hence

$$\frac{P_{it}}{P_t} \equiv P_{it} = \frac{\frac{1}{(\tilde{A}_{it})^{1-\alpha}}}{\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{\tilde{A}_{jt}^{1-\alpha}}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}} = \left(\frac{\tilde{A}_{it}}{\tilde{A}_t}\right)^{-(1-\alpha)}, \tag{17}$$

where the term

$$\tilde{A}_t \equiv \left\{\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\tilde{A}_i^{1-\alpha}}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}\right\}^{-\frac{1}{1-\alpha}} \tag{18}$$

is the *aggregate productivity index* and it will be used for normalisation to obtain stationary variables (see section 3.5). The relative price of intermediate good i can also be found to depend only on own productivity relative to the productivity of the others. The higher is the productivity in a sector relative to the others, the lower must be its relative price. To see the intuition, recall the CES aggregator. A more productive firm finds it profitable to produce more, but consuming more of one type of good decreases its utility. In order to sell more goods, it is necessary to decrease the price. Also note that the interest rate and wage rate cancel out and do not play a role for the relative price. The reason is that they affect both individual price P_i and the aggregate price P in the same way.

3.3.3 Profits and asset prices

The profit of a production firm (and of a recently successful research firm) is

$$\Pi_{it} = P_{it}Y_{it} - W_t l_{it}^P - r_t K_{it},$$

which can be solved to get

$$\Pi_{it} = Y_t \tilde{a}_{it}^{(1-\theta)(\alpha-1)}\left[(1+e_{it})^{1-\alpha} - 1\right], \tag{19}$$

where $\tilde{a}_{it}$ is a measure of the relative productivity of one sector relative to all other sectors and is defined as

$$\tilde{a}_{it} \equiv \frac{\tilde{A}_{it}}{\tilde{A}_t}. \tag{20}$$

The relative productivity turns out to be the crucial variable in the model. Equation (19) shows that as long as $\theta < 1$, the profits are decreasing in own relative productivity $\tilde{a}$. This might seem counter-intuitive at first, but it is the implication of the CES preferences affecting the pricing via the price index. In particular, whenever $\theta < 1$ the goods of different industries are strong complements. A more productive firm finds it profitable to produce more than a less productive firm (in a different industry). However, in order to sell its bigger output, the production firm has to lower its price as the consumers marginal utility declines. However, because of the complementarity, the marginal utility of a good declines faster the stronger the degree of complementarity is.[10]

This means that the complementarity in the consumer utility function generates externalities for production firms and via the expected profits for research firms as well. In particular, conditional on the same value of innovation step e, *if she could choose*, the researcher would rather be in a low productivity industry rather than a higher one. This mechanism balances the model so the relative productivities $\tilde{a}$ have mean reverting behavior.

Equation (19) also reveals that profits are scaled with aggregate output Y and increase with the innovation step e. All else equal, the higher the quality of the research project μ, the bigger is the productivity step e, the higher is the mark-up over marginal costs of the incumbent production firm and hence the higher are the profits a successful research firm will earn.

The quality of innovation is therefore important via two channels. First, for given research effort the likelihood of being successful and hence of getting access to future profits increases with the quality of the project. Second, the better the innovation the higher are these profits. The interplay of these two motives makes the problem significantly non-linear.

Households invest in shares of both production and research firms. For the value of the production firm their optimality condition therefore requires

$$Q_{it}^{P} = \mathsf{E}\left[\bar{\mathbb{1}}_{it} m_{t+1}(\Pi_{it+1} + Q_{it+1}^{P})\right], \tag{21}$$

where Q_{it}^{P} is the value of the production firm in sector i at time t and $\bar{\mathbb{1}}_{it}$ is an indicator function which is equal to one if the production firm survives, i.e. the

[10] A similar mechanism can be found in Acemoglu and Guerrieri (2008).

current research firm is unsuccessful:

$$\bar{\mathbb{1}}_{it} = \begin{cases} 1 & \text{if} \quad \mu_{it} + f\left(l_{it}^R\right) + \varepsilon_{it} \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The value of a research firm, on the other hand, has to satisfy

$$Q_{it}^R = \mathsf{E}\left[\mathbb{1}_{it} m_{t+1}(\Pi_{it+1} + Q_{it+1}^P)\right], \tag{23}$$

where $\mathbb{1}_{it}$ is an indicator function which is equal to one if the research firm is successful (the complement of $\bar{\mathbb{1}}_{it}$):

$$\mathbb{1}_{it} = \begin{cases} 1 & \text{if} \quad \mu_{it} + f\left(l_{it}^R\right) + \varepsilon_{it} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

If the research firm is successful it will become a production firm. Shareholders of this firm will hence get next period's profits and will still own shares in the firm which will then be a production firm of value Q_{it+1}^P. This is reflected in equation (23).

3.3.4 Problem of the research firm

All research firms are fully owned by the households; a research firm initially sells s_{it}^R fraction of its share to household on a market, keeping $1 - s_{it}^R$ to itself. The firm is run by a mutual fund, which is ultimately owned by the households. The fact that different households can hold different firms means that the firms compete on the market and maximise the firm value rather than directly maximising household utility (for example by maximising employment). Thus the choice of research labor $\tilde{l}$ maximises the value of the research firm net of labour costs:

$$\arg\max_{l_{it}^R} \mathsf{E}\left[\mathbb{1}\left(l_{it}^R\right) m_{t+1}\left(\Pi_i\left(l_t^R\right) + Q_{it+1}^P\left(l_{it}^R\right)\right)\right] - l_{it}^R W_t. \tag{25}$$

Note that both the likelihood of being successful as well as future profits and the value of the production firm directly depend on the choice of current research labour. The reason is that all these terms are a function of the innovation step e_{it+1} which depends on research labour (see equation 11).

3.4 Equilibrium

The sequence of events and actions in the model is as follows:

1. First, at the beginning of period t, in all sectors $i = \{1 \dots N\}$ a research firm is born with a research idea of quality μ_{it}. The vector $\boldsymbol{\mu}_t = (\mu_{1t}, \dots, \mu_{Nt})$ summarises the quality of current research projects in all industries and is public knowledge.

2. Second, based on $\boldsymbol{\mu}_t$, as well as all the other state variables, research firms issue stocks and with the proceeds hire workers l_{it}^R to improve their research ideas. Also, households production labor l_{it}^P is hired, output is produced, households decide how much to save and they consume. Note that $\boldsymbol{\mu}_t$ also affects the decisions of the housholds, both directly through asset prices and through growth expectations via the stochastic discount factor.

3. Finally, at the end of the period (after all markets are cleared and consumption took place), in each sector there is an idiosyncratic shock ε_{it} to research, which determines the success of the research project in each sector.

The state vector $\boldsymbol{\Sigma}_t = [K_t, \boldsymbol{e}_t, \boldsymbol{\mu}_t, \tilde{\boldsymbol{a}}_t]$ consists of aggregate capital stock K_t and the mark-ups of currently producing firms $\boldsymbol{e}_t$, the quality of projects of the current research firms $\boldsymbol{\mu}_t$ and the relative productivity of current producers $\tilde{\boldsymbol{a}}_t$. The dimension of the state vector is $3N + 1$.

An equilibrium is a sequence $\{l_t^P, l_t^R, K_{t+1}, r_t, w_t, s_{1t+1}^P, \dots, s_{Nt+1}^P, s_{1t+1}^R, \dots, s_{Nt+1}^R\}_{t=0}^{\infty}$ such that

1. $\{l_t^P + l_t^R, K_{t+1}, s_{1t+1}^P, \dots, s_{Nt+1}^P, s_{1t+1}^R, \dots, s_{Nt+1}^R\}_{t=0}^{\infty}$ solves the household problem

2. $\{l_{it}^R\}_{t=0}^{\infty}$ solves the research firm problem

3. $\{l_{it}^P, K_{it}\}_{t=0}^{\infty}$ solves the production problem of the production firm

4. markets for labour and capital clear

3.5 Transforming the model into stationary form

As the productivity grows, the model economy produces more and more output. While there is no deterministic trend (like in a model where aggregate productivity has an exogenous growth rate, for example an AR(1) process with drift), it is still possible to define the aggregate level of technology and then show that the variables scale linearly with this technological index.

Observing the aggregate price equation (16), it is natural to define aggregate productivity $\tilde{A}_t$ as in equation (18). It follows that if $\tilde{A}_{it} = \tilde{A} \quad \forall i = 1, \ldots, N$, then $\tilde{A}_t = \tilde{A}$. Along the balanced growth path, $\tilde{A}_{it} = \bar{A}_t$, then $\tilde{A}_t = \bar{A}_t$, so aggregate productivity grows together with the growth in the individual industries. It is useful to derive the growth rate of productivity g_{t+1}:

$$g_{t+1} \equiv \frac{\tilde{A}_{t+1}}{\tilde{A}_t} = \frac{\left[\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N}\sum_{i=1}^{N}\tilde{a}_{it}^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}. \tag{26}$$

In particular, equation (26) shows that the change in productivity from one period to the next is a weighted average of innovations e in individual sectors. The innovations which happen in relatively less productive sectors receive bigger weight, because increasing the supply of the most scarce good has a bigger effect than the same change in the good which is already plentiful.

With the aggregate price level normalisation, equation (16) becomes

$$1 = \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{r_t}{\alpha}\right)^{\alpha}\tilde{A}_t^{-(1-\alpha)},$$

which in normalised terms (substituting $w_t = W_t/\tilde{A}_t$) leads to

$$1 = \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{r_t}{\alpha}\right)^{\alpha}. \tag{27}$$

Recall that the shared shape of production function among different industries implied that the same optimal labour/capital ratio is used across all industries (equation (7)). This symmetry allows to construct an aggregate demand for factors of production. Using the aggregation rules, $l_t^P = 1/N\sum_{i=1}^{N} l_{it}^P$ and $K_t = \frac{1}{N}\sum_{i=1}^{N} K_{it}$, the aggregate level relation between capital and labour has to satisfy

$$k_t = l_t^P \frac{w_t}{r_t}\frac{\alpha}{1-\alpha}. \tag{28}$$

The normalised labour supply condition becomes

$$c_t = \frac{\gamma-1}{\phi}\left(1-\left(l_t^P + l_t^R\right)\right) w_t. \tag{29}$$

So far, the effect of the normalisation has been similar to having a fixed trend. However, in my model, the productivity does not fluctuate around some fixed trend, it changes in a step-wise fashion, i.e. it never goes down. This has important implications for normalised capital. Suppose that yesterday the household chose consumption and saving such that, given the expected growth in production, the *normalised* capital today should be at a particular level x. However, suppose that due to a lucky realisation, the growth rate has actually been higher than what was expected yesterday. Given the higher than expected productivity, the *realised* value of *normalised* capital today is *below* x.[11] The absolute level of capital K saved at the end of one period carries to the next period without any change, the normalised capital k is affected by the realised growth and becomes a random variable.

$$k_{t+1} \begin{cases} < k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} > \mathsf{E}_t \tilde{A}_{t+1}, \\ = k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} = \mathsf{E}_t \tilde{A}_{t+1}, \\ > k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} < \mathsf{E}_t \tilde{A}_{t+1}. \end{cases}$$

Normalised equilibrium conditions The system of equations in normalised terms is as follows (for all derivations see appendix A - C):

$$1 = \beta \mathsf{E}\left[\left(g_{t+1}\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{1-(l^P_{t+1}+l^R_{t+1})}{1-(l^P_t+l^R_t)}\right)^{-\phi}(r_{t+1}+1-\delta)\right], \tag{30}$$

$$w_t = (1-\alpha)\left(\frac{k_t}{l^P_t}\right)^{\alpha}, \tag{31}$$

$$r_t = \alpha\left(\frac{l^P_t}{k_t}\right)^{1-\alpha}, \tag{32}$$

$$y_t = \frac{k_t^{\alpha}\left(l^P_t\right)^{1-\alpha}}{\sum_{i=1}^N\left[(1+e_i)\tilde{a}_i^{1-\theta}\right]^{-1+\alpha}}, \tag{33}$$

$$c_t = \frac{\gamma-1}{\phi}\left(1-\left(l^P_t+l^R_t\right)\right)w_t, \tag{34}$$

$$g_{t+1}k_{t+1} = y_t + (1-\delta)k_t - c_t, \tag{35}$$

$$g_{t+1} = \frac{\tilde{A}_{t+1}}{\tilde{A}_t} = \frac{\left[\frac{1}{N}\sum_{i=1}^N[(1+e'_i)\tilde{a}_i]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N}\sum_{i=1}^N\tilde{a}_i^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}, \tag{36}$$

$$\tilde{a}_{it+1} = \frac{(1+e_{it+1})\tilde{a}_{it}}{\left[\frac{1}{N}\sum_{i=1}^N[(1+e_{it+1})\tilde{a}_{it}]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}, \tag{37}$$

$$e_{it+1} = \max\{0, \mu_{it} + \kappa\left(l^R_{it}\right)^{\iota} + \varepsilon_{it}\}, \tag{38}$$

[11] This is equivalent to a one-off unexpected increase in the level of productivity in the Solow growth model.

$$q_i^P = \mathsf{E}\left[\bar{\mathbb{1}}_{it} m_{t+1} g_{t+1} (\pi_{it+1} + q_{it+1}^P)\right], \tag{39}$$

$$q_i^R = \mathsf{E}\left[\mathbb{1}_{it} m_{t+1} g_{t+1} (\pi_{it+1} + q_{it+1}^P)\right]. \tag{40}$$

4 Solution method

In this section I explain the main ideas behind the solution method I use to solve the present model. The combination of three factors makes this problem challenging: first, high dimensionality; second, the fact that innovation has to be *good enough* to be implemented which introduces kinks that propagate through the model; and finally, the fact that the growth rate (which affects the effective rate of discounting) in the model is endogenous. Further information are provided in appendix E, where I also explain the procedure used to obtain the generalised impulse-response functions. Even more detailed description can be found in Rozsypal (2014b).

To solve the model presented in the previous section, I extend the method of ergodic grids developed by Judd *et al.* (2012) to solve for larger models. Furthermore, I allow the shocks to have kinks in the way they affect the outcomes in the model.

The kinks are important because no matter how bad the innovation outcome is, the worst possible change in productivity is zero. At the same, once a certain threshold is reached, the magnitude of the research outcome begins to matter as the size of the impact on future profits, distribution of productivities etc does directly depend on the size of the innovation. While this problem arises in the research firm optimisation, it affects all agents with intertemporal problems as the outcomes of research in this period will affect the state of the economy in the next period.

4.1 Extended projection algorithm

In the setting of the present model, perturbation methods typically do not deliver reliable solution. First, with highly nonlinear models it is not clear what the appropriate steady state should be. Second, even higher order perturbation methods smooth out the effect of the kink.

Therefore, it is necessary to use a global method instead of relying on local approximations. Global methods solution boils down to two tasks. First, it necessary to find a flexible functional form that is capable to approximate the policy function in the model well (traditional example is a function of a polynomial in the state variables, possible combined with some outer link function). Second, it is necessary to select a set of points which capture the area in which the model lives. For one dimensional models, it usually enough to use an equidistant grid around the known steady state. Simulation methods have been developed for situations where it is not apriori known where the grid should lie (den Haan and Marcet, 1990). Judd *et al.* (2012) (JMM hereafter) propose a method that combines sparsity of the simulated grids with the projection based on numerical quadrature. As with any simulation based method, the problem is that a bad initial guess for the policy function might take the model to a particular area of the state space far from the area where the true solution lives. For example, if the initial guess is such that the agents save too much, then the grid will be constructed with higher levels of capital. Ideally, the new solution should update the policy function so the agents save less, and the model needs to be simulated again for the new grid to be constructed. Unfortunately, simulation methods are often fragile to changes in parameters, which requires to make only very small updates in the coefficients of the policy function which significantly increases the number of simulations needed to arive to the correct area of the state space.

To make this procedure faster, I propose to extend the method by JMM by a simple step. JMM propose iteration on household Euler equation to update the policy function. However, at this step it is easy to explore further the implied behaviour. In particular, the consumption/saving decision implied by the policy function also determines the change in the capital stock. For example, if the household is saving at the vast majority of the grid points, then we know that if the simulation is repeated with the new policy function, agents will accumulate more capital and the grid will be moved up.[12] Therefore it is possible to move up the whole grid by a small amount of capital and continue with the error minimisation without the need to loose time by simulating the model. This way, many simulation steps can be saved at the beginning of the solution if our initial guess was imprecise.

4.2 Expectations and kinks

I use Gauss-Hermite numerical quadrature to evaluate the expectations. However, the favourable properties of quadrature rules are guaranteed only for smooth functions. I address the kinks in the following way. First, the location of kinks is easy to compute. Second, the problem of different agents can be exploited so only the part above (or only the part below) a threshold needs to be computed. In particular, for

[12] Capital is typically predetermined variable and hence it is likely to be part of the state space grid.

any function g, the following holds

$$\begin{aligned}\mathsf{E}[g(\boldsymbol{\varepsilon}_t, \boldsymbol{\mu}_{t+1})] = {} & \mathsf{E}[g(\varepsilon_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})|\varepsilon_{it} \leq \underline{\varepsilon}_{it}]\mathsf{P}(\varepsilon_{it} \leq \underline{\varepsilon}_{it}) \\ & + \mathsf{E}[g(\varepsilon_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})|\varepsilon_{it} > \underline{\varepsilon}_{it}]\mathsf{P}(\varepsilon_{it} > \underline{\varepsilon}_{it}).\end{aligned}$$

Using the no effect below the threshold property and $\mathsf{P}(\varepsilon_{it} \leq \underline{\varepsilon}_{it}) = \Phi(\underline{\varepsilon}_{it})$:

$$\mathsf{E}[g(\boldsymbol{\varepsilon}_t, \boldsymbol{\mu}_{t+1})] = \mathsf{E}[g(\underline{\varepsilon}_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})]\Phi(\underline{\varepsilon}_{it}) + \mathsf{E}[g(\varepsilon_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})|\varepsilon_{it} > \underline{\varepsilon}_{it}](1 - \Phi(\underline{\varepsilon}_{it})).$$

Now for example, the production firm does not survive if the realisation of the luck shock ε is above the threshold $\bar{\varepsilon}$ (meaning that $g(\varepsilon_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})|\varepsilon_{it} > \underline{\varepsilon}_{it} = 0)$ and hence only $\mathsf{E}[g(\underline{\varepsilon}_{it}, \boldsymbol{\varepsilon}_{-it}, \boldsymbol{\mu}_{t+1})]$ needs to be evaluated.

5 Calibration

The calibration of the present model poses three challenges which are different to most standard business cycles models. First, the present model demonstrates a mechanism how individual innovations which additively change productivity in different sectors generate growth in aggregate productivity and hence long term growth of output, whereas standard business cycle models are calibrated on stationary data with no relation to the growth rate of the economy. Hence the present model need to generate two set of facts which are traditionally not generated by the same type of models and needs to be evaluated accordingly.

Second, the nonlinear nature of the present model makes the calibration even more challenging. The traditional way how to calibrate deep parameters in linear business cycle models is to obtain analytical formulae for steady state values of some variables and use the long run averages of counterparts in the data to infer the deep parameters. The linearity of such model make the steady state independent of the volatility of shocks. However, given the non-linearity introduced by the innovation implementation this is no longer possible. The endogeneity of research and corresponding growth rate in the model means that the only way how to obtain steady-state values is by simulation.

Finally, given the iteration based solution procedure, the calibration targets can be evaluated from simulation only after the policy rules have converged. Given the time needed for the model to converge for one set of parameters, it is not feasible to implement fully fledged methods of moment estimation for the full set of model parameters.

My calibration strategy is hence the following. First, I set parameters which are standard in the RBC literature $(\alpha, \beta, \gamma, \delta)$ together with a value for θ, to values used in other papers. Using these standard values, I calibrate the other parameters $(\phi, \kappa, \iota, \mu_0, \sigma_\mu, \sigma_\varepsilon)$ to match the growth rate and innovation success rate found in the US data.

The summary of parameters taken from the literature is shown in table 1. Parameter α is typically obtained as the income share of capital. I pick $\alpha = 0.34$, which is located roughly in the middle of values used in the literature.[13] Risk aversion is a controversial parameter as there is vast range of parameters suggested by micro studies (generally below 1) to values around 10 suggested by asset pricing literature. One particularly popular option is to use 1 which simplifies CRRA utility function into logarithm. I use $\gamma = 2$. The depreciation is considered to be 10% annually which gives $\delta = 0.025$. Depreciation together with long term interest rates is used to calibrate the discount rate and I use a standard value $\beta = 0.99$ for quarterly frequency. However, the link between the discount factor, depreciation rate and the interest rates is more complicated in the present model due to the endogenous growth rate.

The value of the complementarity parameter in consumer CES preferences θ plays a crucial role in the proposed mechanism and hence deserves careful calibration. Based on the long-term movements in prices and quantities of different sectors, the structural transformation literature consensus is that the number is lower than one, but the exact magnitude depends on the specific context. Ngai and Pissarides (2008) use 0.3 for a three sector model, Buera and Kaboski (2009) use 0.5 while admitting that the estimated number would be close to zero (page 473, footnote 3) and Acemoglu and Guerrieri (2008) in their two sector model use values between 0.56 and 0.86. Another strand of empirical literature estimates the demand elasticity, Falvey and Gemmell (1996) find elasticity 0.3 for services, Blundell *et al.* (1993) find numbers between 0.5 for food and fuel, to 0.8 for transport (1.5 for alcohol). More recently, Atalay (2014) estimates the complementarity of intermediate good production to be 0.654 in the base line estimation.[14] For their two sector, two countries model, Stockman and Tesar (1995) estimate the elasticity of substitution between tradebles and non-tradebles to be 0.44 using data from 30 countries. Finally, Herrendorf *et al.* (2013) build a three sector model and reconcile different estimates by showing that methods using final expenditure view lead to estimates of 0.8 magnitude, whereas the value added approach leads to a much smaller number 0.002. Confronted with

[13] Others have used values ranging from 0.3 (Greenwood *et al.*, 2000) to 0.36 Jones *et al.* (2005a).

[14] Broda and Weinstein (2006) find higher numbers, but they use products in less coarse definition and hence for each variety, there are much closer substitutes which increases the estimate of substitutability.

this set of existing calibrations, I use an intermediate value of $\theta = 0.5$.

There are two effects determining the behavior of the research firm. First, the value of the production firm the research firm might become and the profits it would earn. Second, conditional on being successful in innovating, the likelihood of being replaced by a future innovator, because it determines the expected number of periods that profits will be earned. It is not clear ex ante which effect will dominate in equilibrium. However, for the relative productivity distribution to be stationary, it is necessary that more research is being done in relative less productive industries, so that the least productive firms catch up rather than the most productive firms becoming even more productive. This means that the parameter θ which governs the degree of complementarity is crucial for ensuring that the research concentrates in the less productive industries. It remains to be determined whether this equilibrating mechanism can be strong enough for the model to be stationary even in situations when the degree of complementarity is weak ($\theta > 1$).

description	label	value
capital share of output	α	0.34
risk aversion	γ	2
capital depreciation	δ	0.025
discount rate	β	0.99
goods complementarity	θ	0.5

Table 1: Model parameters taken from the literature

The primary target for calibration is the growth rate. There are five crucial parameters which directly affect the growth; two parameters affecting the slope (κ) and curvature (ι) of the research production function, and then mean and variance of the quality of research project shock μ and the variance of the luck shock ε. Implementation of only the good projects means that an increase in volatility of either of the two shocks increases the mean growth rate, conditional on keeping the research effort constant. However, increased innovation success rate decreases the value of the production firm by decreasing its expected life-span, whereby decreasing the returns to research. Furthermore, increasing the volatility of the luck shock ε has different implications conditional on the quality of the project μ_{it}. In particular, for very bad projects (μ_{it} very negative), applying reasonable research is not enough to achieve $\mu_{it} + \kappa(l^R)^\iota > 0$. In such a situation, increasing σ_ε increases the expected chance of innovation. However, if the project is very good and $\mu_{it} + \kappa(l^R)^\iota > 0$, then increasing σ_ε increases chances of a very bad ε realisation and ultimately decreases

description	label	value
labour disutility	ϕ	1.33
research production $\kappa\left(l^R\right)^\iota$	ι	0.33
	κ	0.5
mean of innovation quality	μ	-0.275
sd of innovation idea	σ_μ	0.03
sd of luck shock	σ_ε	0.14
number of sectors	N	3

Table 2: Calibrated parameters

the probability of innovation. There is also an important link between the growth rate and the normalised steady state level of capital. Higher growth rate requires households to save more (and consume less) to sustain the same level of normalised capital. However, at the same time, higher growth induces people to consume more and save less via the wealth effect.

The model is sensitive to the choice of research production function parameters and the stochastic properties of the two shocks. To calibrate these, I look also at the implied innovation success rate. Michelacci and Lopez-Salido (2007) have a model where some firms are allowed to innovate to catch up with the frontier. To calibrate this probability, they match plant level productivity data from Baily *et al.* (1992) and propose that the probability of innovation success should be 0.063.[15] This is a challenging target to implement, because the observed growth has to be delivered by an event which happens only in very small fraction of situations.[16] To get such an outcome, the mean of μ shock has to be negative.In order to achieve such a low innovation success rate, I set the mean quality of innovation negative ($\mu = -0.275$). Intuitively, this corresponds to the fact that an average idea is a rather bad one.

The relative productivity dispersion is affected by two factors. First, the degree of complementarity directly regulates the incentives to do research as a function of relative productivity via the demand channel. Secondly, the relative importance of research production function and shocks matter. The higher is the volatility of shocks (both μ and ε), the *relatively* less is the innovation outcome affected by research effort and therefore the higher the dispersion because it is the research effort which is pushing down the dispersion among sectors.

[15]See Michelacci and Lopez-Salido (2007, table 2 and the discussion below).

[16]However, it also shows that linearisation based solution methods would have a hard time to achieve such an outcome unless resorting to continuum of sectors.

Labour disutility ϕ is informative about the hours worked. At the same time, an increase in hours also increases the capital stock, which increases the wage rate which in turn increases wage costs and decreases research. Furthermore, the non-separability of consumption and leisure in the utility function means that any change in the level of leisure also affects the marginal utility of consumption.

Given the aforementioned strong linkages between all parameters, there is no one to one mapping between the parameters and the targeted moments. Targeted moments include growth rate of GDP, level of employment as well as the correlations among the endogenous variables at the business cycle frequencies. Table 2 details the calibrated parameter values and table 3 reports the resulting simulated moments and compares them to the corresponding data moments.[17]

	value	source	model
annual growth rate	0.028	FRED	0.027
innovation success rate	0.063	Michelacci and Lopez-Salido (2007)	0.085
time spent working	0.333	Hansen and Wright (1992)	0.355
Y/K	0.1	Hansen and Wright (1992)	0.306
I/Y	0.171	FRED	0.103
interest rate	0.036	FRED	0.100

Table 3: Targeted moments. The annual growth rate is computed as average of real GDP growth. I/Y is computed as a average ratio between nominal gross private domestic investment and nominal GDP. Interest rate constructed from 3M Treasury bill (average, 1932-2014). Other series are from FRED database and range from 1947Q1 to 2014Q2.

6 Results

The aim of the present model is to generate persistent fluctuations with idiosyncratic iid shocks. Given that the shocks and the mechanism is novel, it is sensible to start simply with the generated time series. Figure 2 shows a simulation of the log of aggregate output over two hundred periods.

The model generates rich dynamics. There are periods of stagnation (approximately between periods 180-220), steady growth (110-170) and even fast growth (0-80). The model is calibrated to quarterly data so each of these episodes would last 10 years.

[17] Jones *et al.* (1993, footnote 2) argue for lower value of time spend working (0.12), Storesletten *et al.* (2011) use 0.3.

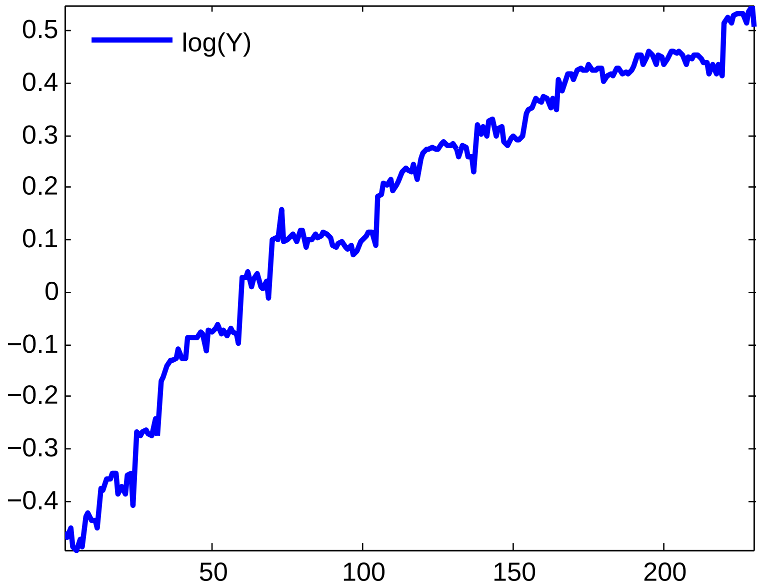


Figure 2: Simulated log of output.

6.1 Research outcomes

The driving force in my model is the endogenous innovation process. Because the policy functions are nonlinear higher order polynomials, the easiest way to visualize them is to plot the implied behavior of changing one state variable at a time.

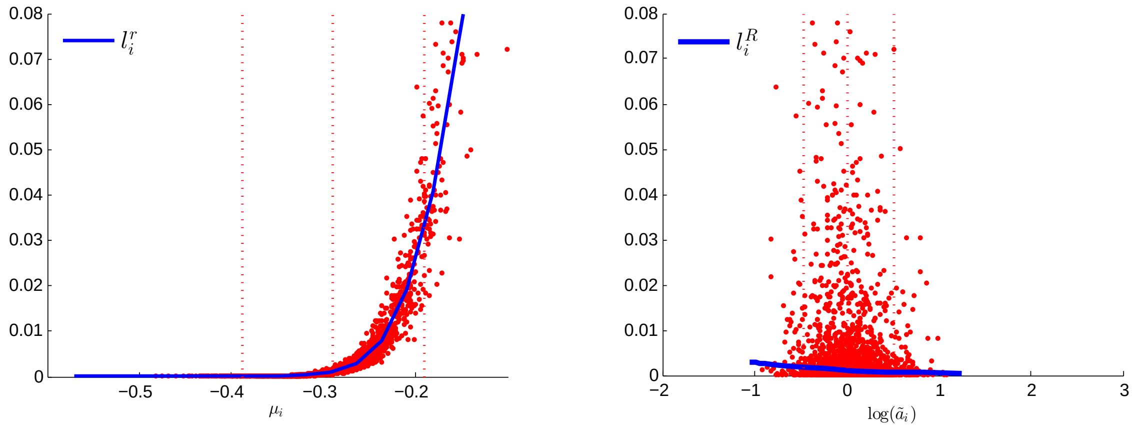
Figure 3 shows the optimal research l^R as a function of the quality of the research project μ and relative productivity $\tilde{a}$. In particular, it is optimal to spend more work on better projects (panel 3(a)). Better projects combined with more research labour also are more likely to be successful (panel 3(c)). The second set of panels shows the effect of relative productivity on research effort and implied probability of successful innovation, more research is done if relatively less productive sectors which also translates into higher probability of innovation.

On the first look it appears that the quality of project (μ) has a potential to affect the outcome much more than the relative productivity; the probability of success for less than average projects is close to zero and for very good projects it is close to 90%, whereas the probability of success in highly relative unproductive sector is only about two percent higher than in the most relatively productive sector. However, the fact that the quality of research project μ is completely transitory whereas the relative productivity is much more permanent means that the smaller effect of the relative productivity is amplified by the fact that is present for long periods of time.

Note that for the average quality of research projects, the probability of succeeding is less than 10%. This is an intended result of the parameterisation which tries to capture the fact that most ideas do not make it into production and lots of start-ups never generate any profits. The deeper point this parametrisation is trying to make is that it is possible that long term growth is generated by a series of small innovations despite the fact that each individual innovator succeeds with only small probability.[18]

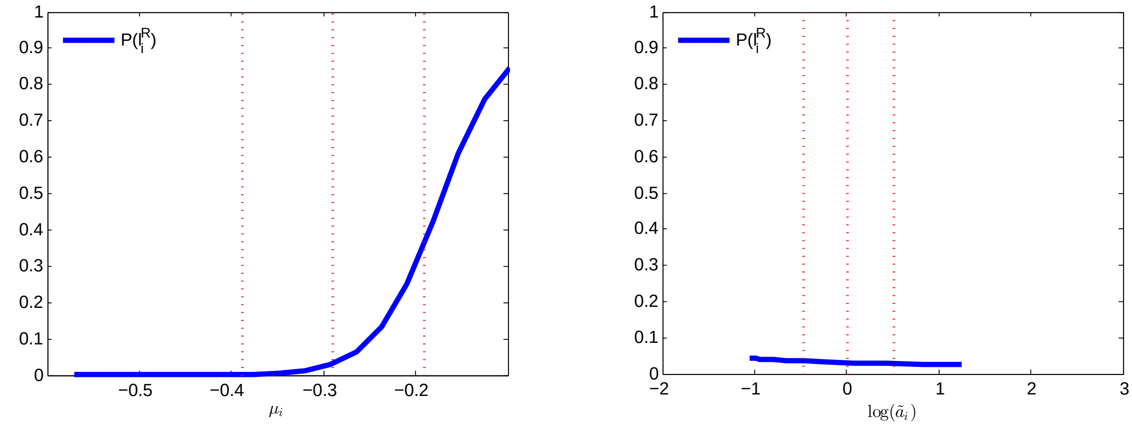
Turning to *relative* productivity (panels 3(c) and 3(d)), more research labour is applied in the sector which is relatively less productive due to the complementarity of production goods: more productive sectors produce more and increase the value of other goods to consumers. The relatively less productive firms hence enjoy a positive externality from increased demand. Therefore, given the same mark-up, it is more profitable to be a producer in a relatively less productive sector.

[18]This fact also highlights the difficulty of standard numerical methods to solve such a model. For instance, steady state typically correspond to a situation where the volatility of shocks is set to zero. In the present model, there would be no growth with volatility of μ and ε set to zero. However, the relationship between growth and research is not trivial. For example, for high levels of volatility, the tenure of production firms is very short which decreases the discounted value of profits they generate. This in turn decreases research as the value of the production firms is smaller.



(a) The optimal amount of research labour in i-th sector as a function of quality of the idea shock μ_i

(b) The optimal amount of research labour in i-th sector as a function of (log) relative productivity $\tilde{a}_i$



(c) Probability on innovation in i-th sector as a function of quality of the idea shock μ_i

(d) Probability on innovation in i-th sector as a function of (log) relative productivity $\tilde{a}_i$

Figure 3: Optimal research l^R and implied probability of innovation. The blue line depicts the optimal research when only one state variable is changing (the quality of the research project μ_i in the left panels and the relative productivity $\tilde{a}_i$ in the right panels) and the red dots show the optimal behaviour from the simulation where all the state variables are changing at the same time. The vertical dotted lines represent 5-th, 50-th and 95-th percentile of μ and $\tilde{a}$ from the simulation.

6.2 Correlations and autocorrelations

Table 4 summarizes the correlations and autocorrelations in the simulated data and the US counterparts.[19] The primary result is that the model generates first order autocorrelation of output of 0.6. Given that there is no autocorrelated shock process, all this persistence is generated endogenously. To shed more light on this results, figure 4 shows the results of the following exercise: simulate the model for 250 periods (to match the number of quarters available in the data), apply HP filter, compute the autocorrelation and plot the resulting histogram. For comparison, the same process is repeated for a simple benchmark RBC model.[20]

	(a) Model					(b) US data				
	y_t	y_{t-1}	c_t	l_t	i_t	y_t	y_{t-1}	c_t	l_t	i_t
	1	0.6	0.5	0.0	0.8	1	0.9	0.7	0.5	0.9
		1	0.3	0.3	0.4		1	0.6	0.4	0.8
			1	-0.6	-0.1			1	0.4	0.6
				1	0.4				1	0.5
					1					1
sd	0.030	0.030	0.020	0.007	0.277	0.002	0.002	0.001	0.007	0.010
rel sd	1	1	0.66	0.23	8.96	1	1	0.85	2.78	5.63

Table 4: Correlations and standard deviation in the simulated data (HP filtered). The numbers in matrix represent correlations, "sd" is standard deviation and "rel sd" is standard deviation relative to sd deviation of output.

The model matched the basic facts (consumption less volatile than output, investment more volatile). However, the model also generates more volatility in the endogenous variables with exeption of labour, which is relatively much less volatile than the counterpart series from the US data.

Figure 4 shows that the RBC model generates slightly higher autocorrelation (mean autocorrelation is 0.7), however, it still does not generate enough to match the data. However, as Cogley and Nason (1995) pointed out, the persistence in the

[19] The data is obtained from FRED database. Time period is 1964Q1-2014Q1. The counterpart variables in the data are: y: Real gross domestic product, l Total private average weekly hours of production and nonsupervisory employees divided by total number of hours in a week, i: Real gross private domestic investment, c: Real personal consumption expenditures. Time series were seasonally adjusted and detrended by HP filter.

[20] For this exercise a canonical RBC model with flexible hours with TFP autocorrelation of 0.95 was used. The full description of this RBC model is in appendix I.

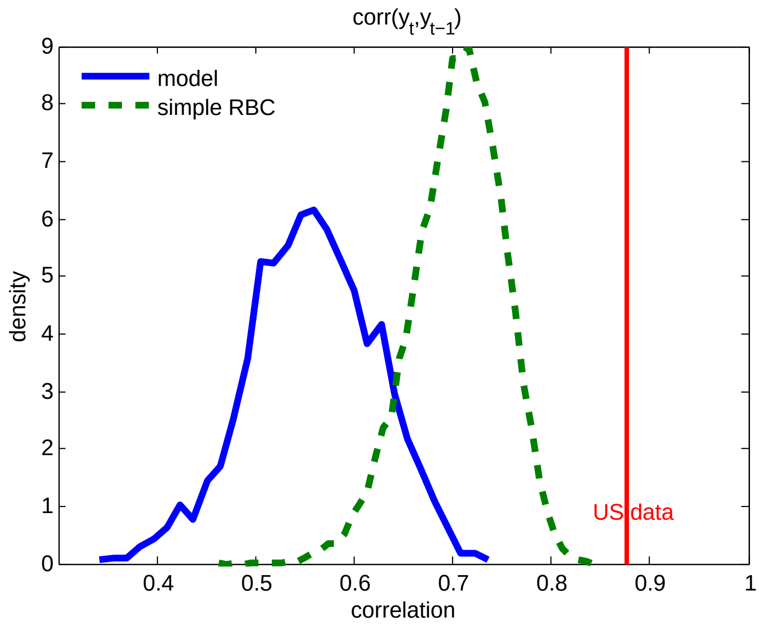


Figure 4: Histogram of simulated autocorrelations of output from the model and a simple RBC model. The vertical line represents the correlation in US data

RBC model is coming from the 0.95 autocorrelation of the underlying productivity process, whereas all the persistence is generated endogenously in the model presented in this paper.

While it is true that the basic RBC model used for comparison is far from the research frontier, any feature which has been found to improve the fit of a simple RBC model with the data (like capital adjustment costs or habit formation) or additional shocks (investment specific productivity, shifts in consumer preferences) can be added to the model in this paper as well. Hence the comparison with the simple RBC model is the most revealing.

The present model does not generate the positive co-movement between consumption and other variables. The reason is that following a good realisation of quality of innovation project μ, there is a shift of labour from production to research. This means that less output is being produced and consumed. Furthermore, if the innovation is successful, then there is a need for capital stock (in absolute levels) to grow to match the new higher level of productivity and during this accumulation period, the consumption is also lower.

Another observation coming from the simulation is that the present model generates more dispersion in correlation than the benchmark RBC model. This is coming from the fact that big innovations have potential to affect the simulated path for much longer periods of time than the mean reverting productivity process in the benchmark RBC. The full set of correlation simulation is in figure 9 in appendix F. The present model performs better than the RBC benchmark in some aspects. In particular, the RBC model generates a clear prediction regarding an almost perfect co-movement between hours and investment. However, this correlation much weaker in the data. In contrast as seen in figure 9, the co-movement generated by the present model is much closer to the one found in the data.

6.3 Spectral analysis

Spectral analysis can provide valuable insights into the cyclical properties of time series which are not immediately apparent by analysing only auto-correlations in the time domain. Figure 5 displays the estimates of the spectrum of real GDP and of simulated output obtained by averaging the periodogram obtained by Fast Fourier Transformation (filtered by HP filter with smoothing $\lambda = 1600$).

The US GDP gap series is depicted by solid blue line, the solid dark green line is the spectrum of time series generated by the model. Finally, the third line (dashed light green) in the plot is the spectrum of a plain vanilla RBC model with productivity represented by AR(1) process with auto-correlation of 0.95. The spectra have been scaled to account for different power of the corresponding time series so this figure aims only at comparison of the relative weight of different frequencies at generating the total variance rather than comparing the total variance itself. The output gap is computed by de-trending using HP filter, the same steps are used for all the three series. The model series can be longer and I thus allow for wider smoothing window.

This result is robust to different calibrations as long as the growth rate generated in the model is reasonably close to the growth rate observed in the data. For calibrations that generate too much growth (over 10% annually), the spectrum flattens as the innovation becomes too frequent.

6.4 Impulse response functions

The impulse-response function are generated as the difference between the control and treatment series from 2400 simulated time paths where the shock hits the first sector in economy in period 20.[21] I show the responses to one and two standard deviation shocks to the quality of innovation project μ and the fact that the latter is not just a scaled-up version of the former demonstrates the strong non-linearity in

[21] Detailed description of the algorithm is provided in appendix G.

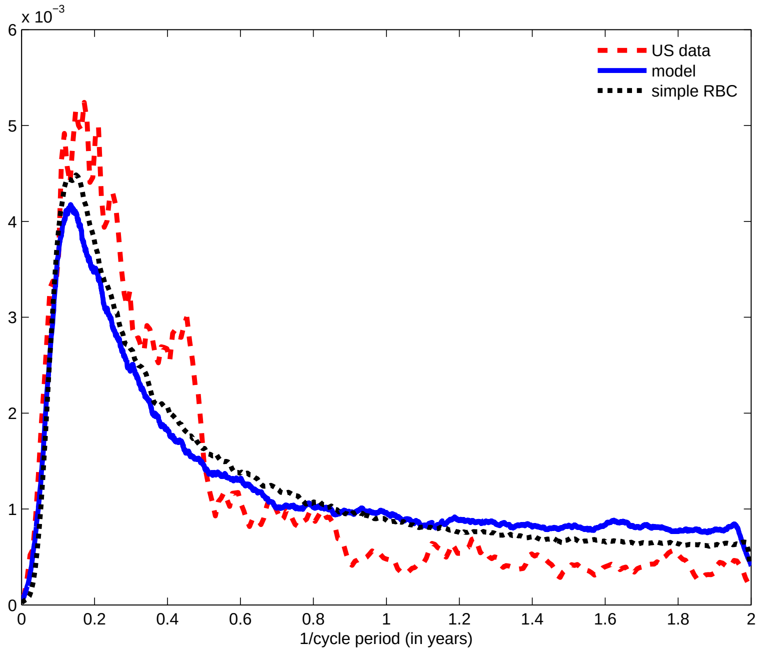


Figure 5: Scaled spectra of the US real output (1947Q1-2013Q4), simulated data from the model and a simple RBC model.

the model. Similarly, the non-linearity of the model also means that the state of the model when the impulse hits affects the outcome. To allow this variation to emerge, I hit the model with the shock only in the period 20, after starting from a state which corresponds to a median over 200 000 period long simulation. Unless stated otherwise, the impulse-response functions are depicted in percentage deviations from the control time series.

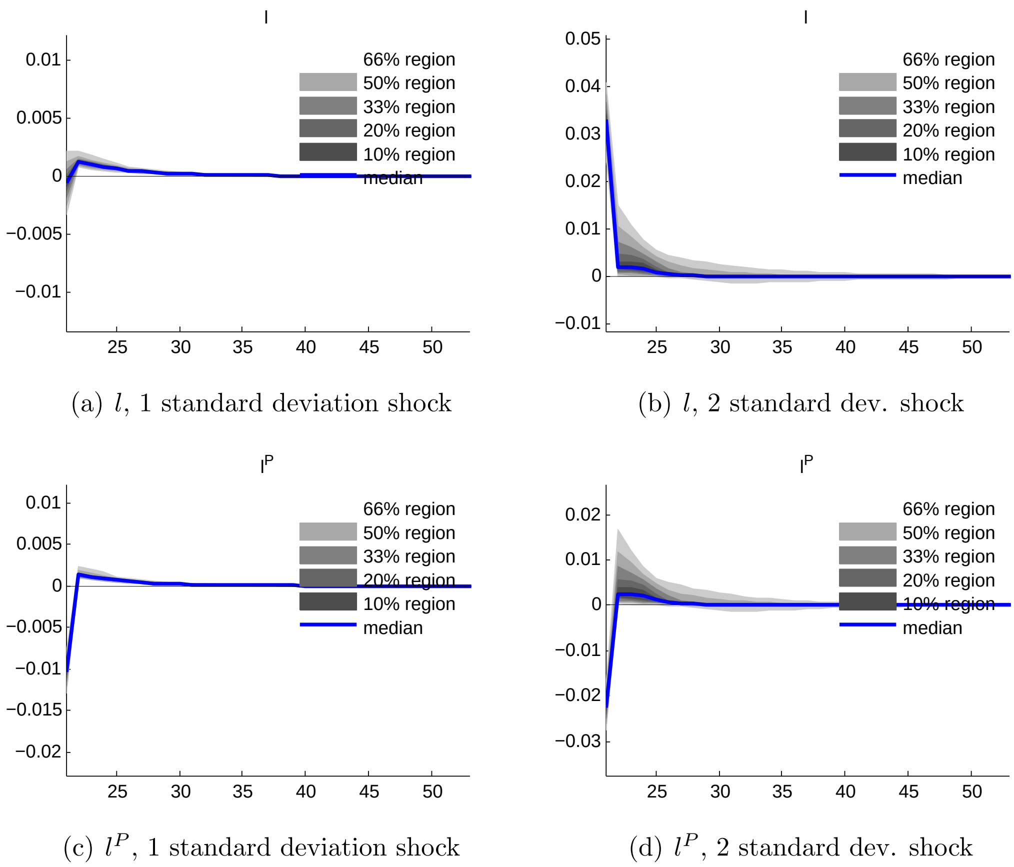
Figure 7 demonstrates the mechanism how the persistence of changes in productivity arises in the model. A favourable research project in the first sector increases the research effort in this respective sector on impact (panels 7(a) and 7(d)). This is effect is drastic in percentage terms because the base is very small (as was already shown by the policy function in figure 3(a)). The labour is shifted from the production firms as l^P falls on impact by 1 (panel 6(c)), or 2 percent respectively (panel 6(d)). This also means that there is less capital accumulation. Nevertheless, the effect on impact on total labour is positive with the magnitude depending on the size of the shock (panels 6(a) and 6(b)).

Median outcome of 1 standard deviations shock is no innovation. Given that the capital fell, there is a long-lasting *negative* effect on the research in all sectors. This is clearly visible in 7(c).[22]

However, for the two deviation shock, the chances of innovation are much bigger. And hence after few periods when the capital is accumulated to be consistent with the new productivity level, there is a persistent positive effect of research in the sectors which were not impacted by the impulse shock (panel).

If successful, this innovation increases the relative productivity and increases incentives to do research in other sectors (equation (19)), which leads to an increase in research in other sectors and ultimately to catching up in relative productivity.

Note also that the standard result in the news shock literature that hours worked fall upon receiving good news about future productivity is not present in this model (measured by the median response). While labour in the production sectors falls, this fall is more than compensated by the increased efforts in the research firms.



(a) l, 1 standard deviation shock

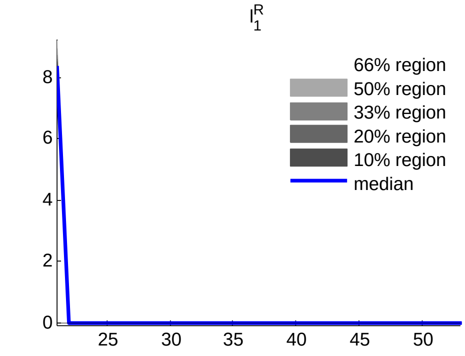
(b) l, 2 standard dev. shock

(c) l^P, 1 standard deviation shock

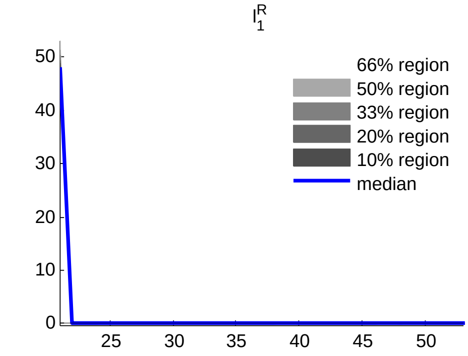
(d) l^P, 2 standard dev. shock

Figure 6: Impulse response functions for response of total and production labour. X axis depicts time (impulse shock hits at $t = 20$), y-axis shows percentage change *relative* to the control series.

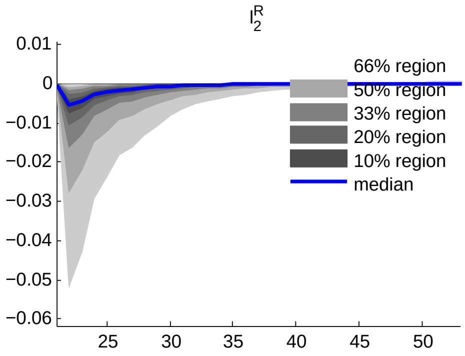
[22]the effect on on the research in the first sector is the same but not visible due to the scaling (the positive effect is orders of magnitude bigger).



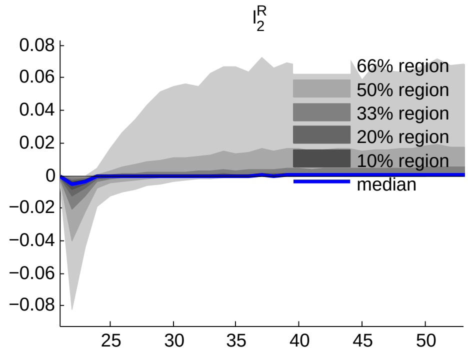
(a) l_1^R, 1 standard deviation shock



(b) l_2^R, 2 standard dev. shock



(c) l_1^R, 1 standard deviations shock



(d) l_2^R, 2 standard dev. shock

Figure 7: Impulse response functions for response of research labour (research in the first (*impacted*) sector and in the second (*not impacted*) sector). X axis depicts time (impulse shock hits at $t = 20$), y-axis shows percentage change *relative* to the control series.

6.5 Estimated TFP vs true changes in productivity

I estimate TFP by assuming an aggregate production function $Y_t = K_t^\alpha (A_t l_t)^{1-\alpha}$ where K is aggregate capital and l is total labour, which in my model corresponds to $l_t^P + l_t^R$. The estimated (logarithm of) TFP is then

$$\widehat{TFP}_t = \frac{\log(Y_t) - \alpha \log(K_t) - (1-\alpha)\log(l_t + \tilde{l}_t)}{1-\alpha}$$

and the true productivity $\tilde{A}$ is defined in (18) on page 18.

The comparison between changes in estimated TFP and the true productivity can be seen in figure 8. First, one can immediately see that the true changes in productivity are only positive, yet the estimated TFP shows periods of technological

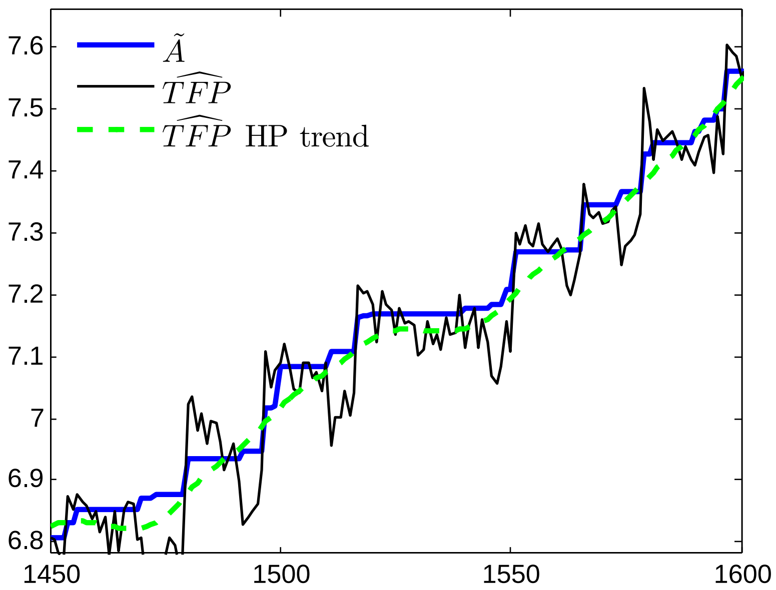


Figure 8: Estimated TFP vs true productivity. To estimate the trend productivity HP filter was used with smoothing parameter $\lambda = 1600$

regress.

In the model, research labour l^R does not contribute to production of output. In periods when l^R is high, this accounting hence overestimates the use of labour in production and hence concludes that the productivity must be low. This means that the volatility of l^R increases the volatility of estimated TFP.

Nevertheless, any filtering method which expects to have about the same number of observations below and above the trend inevitably fails when confronted with series where the changes happen stepwise and the steps are only in one direction.

The knowledge of the true underlying process is hence important for choosing the correct filtering method. Having said that, the correlation between the true and estimated productivities is still high (0.81).

7 Conclusion

It is not controversial to claim that changes in productivity are one of the drivers of economic growth in the long run. If the aggregate productivity is a measure of productivity of individual firms, and if individual firms increase their productivity by innovating, then innovations of individual firms affect the aggregate productivity. Moreover, the growth of aggregate productivity over long periods of time is in fact a result of accumulation of many small innovations on the firm or sectoral level. This paper investigates the short run implications thereof.

To do so, I build an integrated framework where both economic growth and business cycle fluctuations are driven by the same shocks. I do so by introducing research firms into a multi-sector RBC model and show how aggregate productivity depends on the results of innovation processes in each sector.

In the model, a successful innovation decreases the marginal costs of production in a given industry. Strictly speaking this means that innovation describes changes in organization of production, rather than introduction of new product varieties. However, the endogenous growth literature has described the close relationship between models with fixed productivity but increasing number of varieties with models with fixed number of varieties but growing productivity. Hence, when drawing conclusions from the present model, I have in mind a broader definition of research. Research expenditures also should be thought of in a broader sense. For example, Barlevy (2007) notes that up to 40% of research wage bill goes to support staff.

I demonstrate that this setting is capable of matching the aggregate growth while requiring reasonable correlations among the endogenous variables and sensible employment numbers. Furthermore, distinguishing between labour in production firms and research firms creates an avenue how to overcome the classical problem in the news shocks literature, i.e. that positive news shocks cause a fall in employment due to a wealth effect. The model generates Pigou's cycles at least for total employment, if not output and consumption. Also, as demonstrated by the spectrum of simulated output data, the model generates persistent cycles. I also show how estimates of TFP can manifest so called technological regress, i.e. periods of negative productivity growth, while the true productivity process is always bounded by zero.

Regarding the methodology, this paper contributes to the computational economics literature by showing how to solve an endogenous growth model where the balanced growth path is endogenously determined with a global approximation method; I partition the model by separating the decisions of agents regarding the production of goods and innovation and develop an iterative algorithm based on projection methods to obtain mutually consistent policy rules. Finally, I develop a method of how to use Gaussian quadrature to evaluate expectations of variables with kinks. In the present setting, the kinks are introduced by the fact that research can only generate improvements in productivity as innovations with negative effects are simply not introduced.

One of the compelling features of rational expectations business cycles literature is the attempt to face the Lucas critique by building fully micro-founded models. We understand now that a sensible economic policy should not be based on reduced form regularities without appreciating that the agents' behavior is conditional on the policy framework we might be trying to change. For this argument to be credible it is crucial that everything what is assumed to be exogenous is really exogenous. However, is the productivity process really truly exogenous to policy questions we might be interested in?

The assumptions about the productivity process play a crucial role in the properties of our models. Cogley and Nason (1995) argue that the amplification mechanism in business cycles is rather weak and they conclude that *"...in models that rely on intertemporal substitution, capital accumulation, and costs of adjusting the capital stock, output dynamics are nearly the same as impulse dynamics. Hence, these models must rely on impulse dynamics to replicate the dynamics found in U.S. data."* Given this prominent role of the productivity process for business cycle literature, the aim of this paper was to explore how the aggregate productivity process could be micro-founded. Examining the deeper processes that generate the changes in productivity we observe on the aggregate level might be fruitful avenue of future research.

The modern macroeconomic literature has introduced a vast array of possible frictions to make business cycle models fit the data better. The model presented in this paper focused on the productivity process in a frictionless world in order to make the proposed mechanism transparent. Introducing some frictions could be an interesting avenue for future research. In particular, in the model the research firms always gain the financing they need in order to hire researchers, whereas in reality new firms suffer from lack of financing (Sedlacek, 2014). The framework presented in this paper could generate periods of low growth in situations when financing is tight endogenously.

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A Households

The optimality conditions are derived using nominal variables. However, the model does not include any nominal friction, the price level itself does not effect any real allocation (what matters are prices if individual goods *relative* to the aggregate price index). Therefore I choose a path for aggregate price index P_t such that the aggregate price level is constant and equal to $P = 1$. To make the equations in the main body of the paper simpler, I define all the prices in the model already in the real terms. However, for sake of completeness, in the appendix I derive the model fully using nominal prices.

A.1 Optimal allocation of Consumption Expenditures

Let's assume that the consumption goods are complements with elasticity θ. The household problem of maximizing utility can be solved using duality approach by minimizing expenditures given some budget Z:

$$\max \left[\frac{1}{N}\sum_{i=1}^{N} Y_i^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \qquad \text{s.t.} \sum_{i=1}^{N} P_i Y_i = Z,$$

combining this FOC for two different goods, I get $\frac{Y_j^{-\frac{1}{\theta}}}{P_j} = \frac{Y_i^{-\frac{1}{\theta}}}{P_i}$ which can be rearranged into

$$Y_i = Y_j \left(\frac{P_i}{P_j}\right)^{-\theta}. \tag{41}$$

Now, using the budget constraint $\sum_{i=1}^{N} P_i Y_i = Z$, I get $Z = \sum_{i=1}^{N} P_i Y_j \left(\frac{P_i}{P_j}\right)^{-\theta} = Y_j P_j^{\theta} \sum_{i=1}^{N} P_i^{1-\theta}$ this solved for Y_j gives $Y_j = \frac{\frac{1}{N}Z}{\left(\frac{1}{N}\sum_{i=1}^{N} P_i^{1-\theta}\right)^{\frac{1}{1-\theta}}} \frac{P_j^{-\theta}}{\left(\frac{1}{N}\sum_{i=1}^{N} P_i^{1-\theta}\right)^{\frac{-\theta}{1-\theta}}}$ and because the price index has form of $P = \left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{1}{1-\theta}}$, the last result can be written as $Y_i = \left(\frac{P_i}{P}\right)^{-\theta} \frac{Z}{P}\frac{1}{N}$ $Y = \left(\frac{1}{N}\sum_{i=1}^{N} Y_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} = \frac{Z}{P}\frac{1}{P^{-\theta}}\frac{1}{N}P^{-\theta} = \frac{Z}{P}\frac{1}{N}$ and hence $PY = \frac{Z}{N}$. Combining this results with (42) I get the relative demand of particular good C_i given the aggregate consumption Y and relative prices $\frac{P_i}{P}$:

$$Y_i = \left(\frac{P_i}{P}\right)^{-\theta} Y \tag{42}$$

A.2 Household budget constraint

The budget constraint is

$$
\begin{aligned}
&P_t(C_t + K_{t+1} - (1-\delta)K_t) \\
= \frac{1}{N}\left[r_t \sum_{i=1}^{N} K_{it} + W_t \sum_{i=1}^{N}(l_{it}^P + l_{it}^R) + \sum_{i=1}^{N} s_{it}^P(\Pi_{it} + Q_{it}^P) - \sum_{i=1}^{N}\left(s_{it+1}^P Q_{it}^P + s_{it+1}^R Q_{it}^R\right)\right]&,
\end{aligned} \tag{43}
$$

which can be re-written as

$$
\begin{aligned}
C_t + K_{t+1} - (1-\delta)K_t &= \frac{r_t}{P_t}\frac{1}{N}\sum_{i=1}^{N} K_{it} + \frac{W_t}{P_t}\frac{1}{N}\sum_{i=1}^{N}(l_{it}^P + l_{it}^R) \\
&\quad + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right) - \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P \frac{Q_{it}^P}{P_t} + s_{it+1}^R \frac{Q_{it}^R}{P_t}\right) \\
&= \frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t + l_t^R) \\
&\quad + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right) - \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P \frac{Q_{it}^P}{P_t} + s_{it+1}^R \frac{Q_{it}^R}{P_t}\right).
\end{aligned}
$$

A.3 Consumer optimality conditions

The Lagrangian is

$$
\begin{aligned}
\mathscr{L} = \sum_{t=0}^{\infty}\beta^t \Bigg[u(C_t, l_t, l_t^R) - \lambda_t\left(C_t + K_{t+1} - (1-\delta)K_t + \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P \frac{Q_{it}^P}{P_t} + s_{it+1}^R \frac{Q_{it}^R}{P_t}\right)\right) \\
+\lambda_t\left(\frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t^P + l_t^R) + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right)\right)\Bigg].
\end{aligned}
$$

The first order conditions are (note the different treatment of l_t^P and l_{it}^R, the former is the aggregate labour whereas the latter is one industry only)

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial Y_t} &= \beta^t\left[u_{C_t} - \lambda_t\right] = 0 \\
&\Rightarrow u_{C_t} = \lambda_t
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial l_t^P} &= \beta^t\left[u_{l_t^P} + \lambda_t W_t\right] = 0 \\
&\Rightarrow \frac{u_{l_t}}{W_t} = -\lambda
\end{aligned} \tag{45}
$$

$$\frac{\partial \mathscr{L}}{\partial l^R} = \beta^t \left[u_{l_t^R} + \lambda_t W_t \right] = 0$$

$$\Rightarrow \frac{u_{l_t^R}}{W_t} = -\lambda_t \tag{46}$$

$$\frac{\partial \mathscr{L}}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left(1 - \delta + \frac{r_{t+1}}{P_{t+1}} \right) = 0$$

$$\Rightarrow \beta \lambda_{t+1} \left(1 - \delta + \frac{r_{t+1}}{P_{t+1}} \right) = \lambda_t \tag{47}$$

$$\frac{\partial \mathscr{L}}{\partial s_{it+1}^P} = -\frac{1}{N} \beta^t \lambda_t \frac{Q_{it}^P}{P_t} + \frac{1}{N} \beta^{t+1} \lambda_{t+1} \bar{\mathbb{1}}_{it} \frac{\Pi_{it+1} + Q_{it+1}^P}{P_{t+1}}$$

$$\Rightarrow \beta \lambda_{t+1} \frac{P_t}{P_{t+1}} \frac{\bar{\mathbb{1}}_{it} (\Pi_{it+1} + Q_{it+1}^P)}{Q_{it}^P} = \lambda_t \tag{48}$$

$$\frac{\partial \mathscr{L}}{\partial s_{it+1}^R} = -\frac{1}{N} \beta^t \lambda_t \frac{Q_{it}^R}{P_t} + \frac{1}{N} \beta^{t+1} \lambda_{t+1} \mathbb{1}_{it} \frac{\Pi_{it+1} + Q_{it+1}^P}{P_{t+1}}$$

$$\Rightarrow \beta \lambda_{t+1} \frac{P_t}{P_{t+1}} \frac{\mathbb{1}_{it} (\Pi_{it+1} + Q_{it+1}^P)}{Q_{it}^R} = \lambda_t. \tag{49}$$

B Firms

B.1 Deriving factor demands by firms

B.1.1 Marginal costs

Given W, r, Y, the problem is to find $\min_{l,K} \quad Wl + rK$ such that $K^\alpha (Al)^{1-\alpha} = Y$

Defining lagrangian $\mathscr{L} = wl + rK + \lambda \left(y - k^\alpha (Al)^{1-\alpha} \right)$ and combining the first order conditions with respect to l, K and λ, we can obtain obtain the relation between the factor inputs

$$K = l^P \frac{W}{r} \frac{\alpha}{1-\alpha}, \tag*{((28), page 22)}$$

Use this result in the production function generates the solutions for the factor inputs

$$l^P(y) = Y \left(\frac{W}{r} \frac{\alpha}{1-\alpha} \right)^{-\alpha} A^{-1+\alpha}, \tag{9}$$

$$K(Y) = Y \left(\frac{W}{r} \frac{\alpha}{1-\alpha} \right)^{1-\alpha} A^{\alpha-1}. \tag{10}$$

The minimized total costs of producing output Y with r, W, A is $\psi(Y) = Wl^P(Y) +$

$rK(Y)$, so the marginal costs are

$$MC(y) = \psi' = Wl^{P'}(Y) + rK'(Y) = A^{\alpha-1}W^{1-\alpha}r^{\alpha}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right]. \tag{8}$$

B.1.2 Aggregate implications of individual production firm behaviour derivations

Now I derive the aggregate output as a function of aggregate factor inputs. Note that all production firms have in general different productivities and the aggregate production function is then a weighted average of individual production functions. Here I derive the weights.

Using the supply side condition on optimal capital/labour ratio in production captured by equation (28), summing it over all industries, dividing by N and normalising by $\tilde{A}_t$ gives

$$k_t = l_t^P \frac{w_t}{r_t} \frac{\alpha}{1-\alpha}. \tag{50}$$

With the aggregate price level normalisation, equation (16) becomes $1 = W_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\tilde{A}_t^{-(1-\alpha)}$ which leads to $1 = w_t^{1-\alpha} r_t^{\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ this combined with the previous results gives $w_t = (1-\alpha)\left(\frac{k_t}{l_t^P}\right)^{\alpha}$ and $r_t = \alpha\left(\frac{l_t^P}{k_t}\right)^{1-\alpha}$ which is the same as in other RBC models with perfect competition.

Finally, after some algebra the relative price can be found to be

$$\frac{P_i}{P_t} = \tilde{A}_{it}^{\alpha-1}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\tilde{A}_i^{1-\alpha}}\right)^{1-\theta}\right]^{\frac{1}{\theta-1}} = \tilde{a}_{it}^{\alpha-1}. \tag{51}$$

From the CES demand, it follows that the demand for output follows $Y_{it} = Y_t\left(\frac{P_{it}}{P_t}\right)^{-\theta}$ so the normalisation and the application of relative price result gives $y_{it} = y_t\tilde{a}_{it}^{\theta(1-\alpha)}$ The normalised production function of individual production firm is $Y_{it} = (1+e_{it})^{1-\alpha}k_{it}^{\alpha}\left(\tilde{a}_{it}l_{it}^P\right)^{1-\alpha}$, which can be solved for capital and combining it with the solution for capital from the optimal capital/labour $k_{it} = l_{it}^P\frac{w_t}{r_t}\frac{\alpha}{1-\alpha}$finally leads to

$$y_t = \frac{k_t^{\alpha}\left(l_t^P\right)^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}}.$$

In the symmetric case ($\tilde{a}_i = 1$ and $e_i = \bar{e}$, $\forall i = 1 \ldots, N$), I get

$$y_t = k_t^{\alpha} \left[(1+\bar{e}) l_t^P \right]^{1-\alpha}$$

and $A_{it} = (1 + e_{it})\tilde{A}_{it}$ so this would correspond to standard labour augmenting production function. Higher innovation step $\bar{e}$ hence increases the output.

B.2 Comparison to monopolistic pricing

The monopolistic price would be computed as the solution to the following problem: $\max_y p(y)y - \psi(y)$. The first order condition is $p'(y)y + p = \psi'$, which can be rewritten as

$$\psi' = p\left(p'\frac{y}{p} + 1\right) = p\left(\frac{\frac{\partial p}{\partial y}}{\frac{p}{y}} + 1\right) = p\left(\frac{1}{\theta} + 1\right) = p\frac{\theta+1}{\theta} = p\frac{1}{\mu},$$

because $\frac{\mathrm{d}p}{\mathrm{d}y}/\frac{p}{y}$ is the price elasticity of demand. This leads to a standard mark-up result; the price a monopolist charges is a fixed mark-up over the marginal costs $p = \mu\psi'$.

If the new firm was to charge the monopolist price which would be above the limit price, then the incumbent firm could enter the market and realise profits. However, if the monopolist price is *lower* than the limit pricing price, then the new firm would maximize its profits charging the monopoly price, as at this price the incumbent could not enter the market anyway. In algebraic terms, the assumption requires that $\mu = \frac{\theta+1}{\theta} > (1+e_i)^{1-\alpha}$, or that the technological growth is *slow enough* so the new entrant finds it profitable to charge a limit price and drive out the incumbent rather than to apply the monopolistic price.

This implies that while the production firms are monopolists, due to the presence of a potential competitor who, despite being less productive, can potentially enter the market, the price mark-ups are smaller than in standard Calvo-type new keynesian models. This means that the prices are lower and the output is higher.

C Equilibrium conditions

Here I solve for the Euler equation in terms of the state variables and the policy functions l^P and l^R. The sequence is following:

1. solve for w,

$$w_t = (1-\alpha)\left(\frac{k_t}{l_t^P}\right)^{\alpha},$$

2. solve for c, normalised household first order labour supply condition

$$\begin{aligned} c_t &= \frac{\gamma-1}{\phi}(1-(l_t^P+l_t^R))w_t \\ &= \frac{(1-\alpha)(\gamma-1)}{\phi}\left(1-\left(l_t^P+l_t^R\right)\right)\left(\frac{k_t}{l_t^P}\right)^{\alpha}, \end{aligned}$$

this can be re-written as

$$c_t = k_t^{\alpha}\frac{(1-\alpha)(\gamma-1)}{\phi}\left(\frac{1-\left(l_t^P+l_t^R\right)}{\left(l_t^P\right)^{\alpha}}\right),$$

and this implies that

$$\frac{\partial c_t}{\partial l_t^P} = k_t^{\alpha}\frac{(1-\alpha)(\gamma-1)}{\phi}\left(-\frac{1+\alpha(1-l_t^P-l_t^R)\left(l_t^P\right)^{-1}}{\left(l_t^P\right)^{\alpha}}\right) < 0.$$

This means that people work more when they consume less, i.e. leissure and consumption are substitutes.

3. solve for y,

$$y_t = \frac{k_t^{\alpha}\left(l_t^P\right)^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}},$$

4. solve for k_{t+1}, using the fact that in equilibrium all the output goes to the household:

$$Y_t = \frac{1}{N}\left[r_t\sum_{i=1}^{N}K_{it} + W_t\sum_{i=1}^{N}l_{it} + \sum_{i=1}^{N}s_{it}^P\Pi_{it}\right],$$

and the fact that the research firm spends all raised equity on labour,

$$W_t l_{it}^R = Q_{it}^R, \qquad \forall i = 1,\ldots,N$$

and the fact that the equilibrium holding of the stocks of the production firm is fixed $s_{it}^P = s_{it+1}^P$ (so there is no income from selling or expenditures from buying more production stocks), leads to

$$k_{t+1} = \frac{1}{g_{t+1}}(y_t + (1-\delta)k_t - c_t),$$

where

$$g_{t+1} = \frac{\tilde{A}_{t+1}}{\tilde{A}_t} = \frac{\left[\frac{1}{N}\sum_{i=1}^{N}[(1+e_{it+1})\tilde{a}_{it}]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N}\sum_{i=1}^{N}\tilde{a}_{it}^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}$$
$$= \left[\frac{\sum_{i=1}^{N}[(1+e_{it+1})\tilde{a}_{it}]^{(\alpha-1)(1-\theta)}}{\sum_{i=1}^{N}\tilde{a}_{it}^{(\alpha-1)(1-\theta)}}\right]^{\frac{1}{(\alpha-1)(1-\theta)}},$$

hence

$$k_{t+1} = \left[\frac{\sum_{i=1}^{N}[(1+e_{it+1})\tilde{a}_{it}]^{(\alpha-1)(1-\theta)}}{\sum_{i=1}^{N}\tilde{a}_{it}^{(\alpha-1)(1-\theta)}}\right]^{\frac{1}{(\alpha-1)(1-\theta)}} \left\{ \frac{k_t^{\alpha}\left(l_t^P\right)^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}} + (1-\delta)k_t \right.$$
$$\left. - \frac{(1-\alpha)(\gamma-1)}{\phi}(1-(l_t^P+l_t^R))\left(\frac{k_t}{l_t}\right)^{\alpha} \right\}.$$

D Proofs

D.1 Proof of result 2

Here I formally show that for a price lower than the limit price, it is profitable to produce more as long $\theta \in (0,1)$. To simplify the algebra, I solve for the optimal price p_1 and quantity y_1 in sector 1 and the price p_2 and quantity y_2 in any other sector 2 are fixed and the prices of input are taken as given.

Proof. If π_1 is below the limit price, then the new entrant is serving the market alone. According to equation (41), the price given output y_1 is equal to $p_1 = p_2\left(\frac{y_1}{y_2}\right)^{-1/\theta}$. The profit function is $pi_1 = p_1 y_1 - w l_1^P - r k_1$. After solving the cost minimisation problem and substituting in the price, the profits turn out to be

$$\pi_1 = p_2\left(\frac{y_1}{y_2}\right)^{\frac{\theta-1}{\theta}} - y_1\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha} A_1^{\alpha-1}.$$

The problem to maximize profits does not have a solution for $y_1 \geq 0$. □

D.2 Proof of result 1

I will show formally that this is the case in a simpler two sector, two period, social planner framework. For analytic convenience, I further assume that there is no capital and the innovation step is fixed at $\bar{e}$ (and increase in research increases the chances of innovation but not its size). I also assume simpler utility function where every labour dis-utility enter additively and the utility of consumption is logarithmic. Since the absolute level of productivity does not matter, I normalise $\tilde{A}_2 = 1$ and define $\lambda = \tilde{A}_1/\tilde{A}_2 = \tilde{A}_1$

In this setting, the production labour decision is trivial, it can be shown that the problem to maximize discounted utility simplifies to the problem of finding optimal amounts of research in the first period, weighting the likely benefits in increased consumption in the second period:

$$u(r_1, r_2, \lambda, \theta) = \beta l^* \mathsf{E}\left[\log Z\right] - r_1^{\psi} - r_2^{\psi}, \tag{52}$$

where $Z = \log\left((1+e_1)^{\theta-1}\lambda^{\theta-1} + (1+e_2)^{\theta-1}\right)^{\frac{1}{\theta-1}}$ is a measure of productivity and $e_i = \bar{e}$ if the innovation in i-th sector is successful and $e_i = 0$ otherwise.

The two crucial parameters are θ, the degree of complementarity and λ, the relative productivity. The parameter λ also captures the dispersion in productivity; the dispersion is the smallest for $\lambda = 1$ and it increases as λ gets both bigger and smaller than 1. Let's define $\{r_1^*(\theta,\lambda), r_2^*(\theta,\lambda)\}$ as the solution the the above problem and $u_{x,y} = \frac{\partial^2 u}{\partial x \partial y}$.

I am going to use the fact that differentiation is a linear operator and the implicit function theorem. Due to properties of differentiation $\frac{\mathrm{d}[f(x)+g(x)]}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{\mathrm{d}g(x)}{\mathrm{d}x}$ and hence $\frac{\partial[r_1^*(\theta,\lambda)+r_2^*(\theta,\lambda)]}{\partial\lambda} = \frac{\partial r_1^*(\theta,\lambda)}{\partial\lambda} + \frac{\partial r_2^*(\theta,\lambda)}{\partial\lambda}$ The implicit function theorem shows that if $h(\boldsymbol{x},\boldsymbol{\gamma}) : \mathbb{R}^{N+M} \to \mathbb{R}^N$ and $h(\boldsymbol{x},\boldsymbol{\gamma}) \equiv \mathbf{0}$ implicitly defines $\boldsymbol{x}(\boldsymbol{\gamma})$ as a functions of parameters $\boldsymbol{\gamma}$ (where $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{\gamma} \in \mathbb{R}^M$), then $\frac{\partial \boldsymbol{x}(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}'} = -\left[\frac{\partial h(\boldsymbol{x},\boldsymbol{\gamma})}{\partial \boldsymbol{x}'}\right]^{-1} \frac{\partial h(\boldsymbol{x},\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}'}$ In my setting, $h(\boldsymbol{x},\boldsymbol{\gamma}) \equiv \mathbf{0}$ is defined by the first order conditions, and therefore

$$\frac{\partial r_1^*(\theta,\lambda)}{\partial\lambda} + \frac{\partial r_2^*(\theta,\lambda)}{\partial\lambda} = -\frac{u_{r_1,\lambda}(u_{r_2,r_2} - u_{r_1,r_2}) + u_{r_2,\lambda}(u_{r_1,r_1} - u_{r_1,r_2})}{u_{r_1,r_1}u_{r_2,r_2} - u_{r_1,r_2}^2}. \tag{53}$$

I am going to show that the right hand side of equation (53) is positive by showing that the denominator is positive and the nominator is negative.

Lemma 1. $u_{r_1,r_1}u_{r_2,r_2} - u_{r_1,r_2}^2 > 0.$

Proof. For (r_1, r_2) to be an interior point solution of the maximisation problem, the matrix $\frac{\partial^2 u(r_1,r_2,\theta,\lambda)}{\partial \boldsymbol{r} \partial \boldsymbol{r}'}$ has to be negative definite. However, if that's the case, then all the eigenvalues are negative. The determinant is 2x2 matrix is equal to the product of its eigenvalues. In two dimensions, the product of two negative numbers is a positive number and hence

$$-u^2_{r_1,r_2} + u_{r_1,r_1}u_{r_2,r_2} = \left|\frac{\partial^2 u(r_1, r_2, \theta, \lambda)}{\partial \boldsymbol{r} \partial \boldsymbol{r'}}\right| > 0.$$

□

In equation (52), the expectations term can be written as

$$\mathsf{E} \log Z = \mathsf{P}^{11}\chi^{11} + \mathsf{P}^{10}\chi^{10} + \mathsf{P}^{01}\chi^{01} + \mathsf{P}^{00}\chi^{00},$$

where χ^{11} is the productivity in a state where both sectors have an innovation, χ^{10} only the first one has an innovation, and P denotes the joint probability of corresponding state. The important fact established here is that r_1 and r_2 affect only P terms not χ terms, whereas λ and θ affect only χ terms and not P's.

There are two inputs into the research production function, *quality of the project* μ_i and *research effort* r_i. In this proof I consider only symmetric situations where $\mu_1 = \mu_2$ and hence the different choices for r_1 and r_2 come only from λ. The research function links the effort in research with the probability of successful outcome. Simplifying the main model, $\mathsf{P}_i = \Phi(\mu_i + r_i)$, as the innovation is successful only if the *luck* shock ε is good enough: $\mathsf{P}^1 = \mathsf{P}(\varepsilon_1 > -(\mu_1 + r_1)) = 1 - \Phi(-(\mu_1 + r_1)) = \Phi(\mu_1 + r_1)$.

Using notation $\Phi(\mu_1 + r_1) = \Phi_1$ and $\varphi(\mu_1 + r_1) = \varphi_1$, I have $\frac{\partial \mathsf{E} \log Z}{\partial r_1} = \varphi_1 \xi_1$ and $\frac{\partial \mathsf{E} \log Z}{\partial r_2} = \varphi_2 \xi_2$, where $\xi_1 = \Phi_2(\chi^{11} - \chi^{01}) + (1 - \Phi_2)(\chi^{10} - \chi^{00})$ and $\xi_2 = \Phi_1(\chi^{11} - \chi^{10}) + (1 - \Phi_1)(\chi^{01} - \chi^{00})$. These two terms represent the benefit of innovation in a given industry on the output in the second period. Successful innovation in the first sector increases λ, whereas successful innovation in the second sector decreases λ. In either case, the benefits of innovation in either sector are always positive, regardless of λ or degree of complementarity.[23]

Lemma 2. *For any relative productivity λ and as long as the complementarity is strong enough ($\theta \in (0, 1)$):*

$$\frac{\partial \xi_1}{\partial \lambda} < 0 < \frac{\partial \xi_2}{\partial \lambda}.$$

This implies that the benefits of innovation are falling with relative productivity of a respective sector.

[23]Even with Leontief preferences, increasing productivity in the relatively more productive sector allows shifting more labour from this sector into the less productive whereby producing more goods.

Lemma 3. *For $\theta \in (0,1)$, we have*

$$\frac{\partial \chi^{11} - \chi^{10} - \chi^{01} + \chi^{00}}{\partial \lambda} = \begin{cases} > 0 & for \quad \lambda < 1 \\ = 0 & for \quad \lambda = 1 \\ < 0 & for \quad \lambda > 1. \end{cases}$$

Proof.

$$\frac{\partial \chi^{11} - \chi^{10} - \chi^{01} + \chi^{00}}{\partial \lambda} = \frac{\overbrace{\lambda^{\theta-2}}^{+}\overbrace{\left(1-(1+\bar{e})^{\theta-1}\right)^2}^{+}\left(\lambda^{\theta-1}-1\right)}{\underbrace{\left(\lambda^{\theta-1}+1\right)\left((1+\bar{e})^{\theta-1}\lambda^{\theta-1}+1\right)\left(\lambda^{\theta-1}+(1+\bar{e})^{\theta-1}\right)}_{+}}$$

□

Lemma 4. *For $\lambda > 1$ but close to 1, $\theta \in (0,1)$, $\mu_1 = \mu_2$, $\psi < 2$ and e small, we have*

$$u_{r_1,r_1} < u_{r_2,r_2} < -u_{r_1,r_2} < 0 < u_{r_1,r_2} < |u_{r_2,r_2}| < |u_{r_1,r_1}|.$$

Proof. For u_{r_1,r_1} and u_{r_2,r_2} all symbols represent positive numbers as long as $\psi > 1$. For u_{r_1,r_2}, all symbols are again positive and it can be shown by direct manipulation that $\chi^{11} - \chi^{10} - \chi^{01} + \chi^{00} > 0$.

For λ close to 1 and $\mu_1 = \mu_2$, the incentives to do research in either sector are similar, hence r_1 is close to r_2. Let's define $f(r) = -r\beta l^* \varphi_1 \xi_1 - \psi(\psi-1)r^{\psi-2}$. For $\psi < 2$, $f(r)$ is negative increasing concave function. Hence $r_1 < r_2 \Rightarrow f(r_1) < f(r_2)$. Because $f(r_i) = u_{r_i,r_i}$, we have $u_{r_1,r_1} < u_{r_2,r_1} < 0 < |u_{r_2,r_1}| < |u_{r_1,r_1}|$.

From lemma 1 we know that $u_{r_1,r_1}u_{r_2,r_2} - u^2_{r_1,r_2} > 0$. But if u_{r_1,r_1} is arbitrarily close to u_{r_2,r_2}, then both u_{r_1,r_1} and u_{r_2,r_2} have to be bigger in absolute value than u_{r_1,r_2}. Finally, looking at the hessian we know that $0 < u_{r_1,r_2}$. □

Lemma 5. *The sign of total change in innovation benefit depends on the likelihood of success in the innovation and if the total chances of innovation are high enough ($1 > \Phi_1 + \Phi_2$), then*

$$\frac{\partial^2 u}{\partial r_1 \partial \lambda} + \frac{\partial^2 u}{\partial r_2 \partial \lambda} < 0.$$

Proof. First, by some algebra we get $\frac{\partial \xi_1 + \xi_2}{\partial \lambda} = -(1 - \Phi_1 - \Phi_2)\frac{\partial \chi^{11} - \chi^{10} - \chi^{01} + \chi^{00}}{\partial \lambda}$. Second, the task is to isolate $\begin{bmatrix} \frac{\partial \xi_1}{\partial \lambda} & \frac{\partial \xi_2}{\partial \lambda} \end{bmatrix}'$ in the following expression: $\begin{bmatrix} \frac{\partial^2 u}{\partial r_1 \partial \lambda} & \frac{\partial^2 u}{\partial r_2 \partial \lambda} \end{bmatrix}' =$

$\beta l^* \left[\varphi_1 \frac{\partial \xi_1}{\partial \lambda} \quad \varphi_2 \frac{\partial \xi_2}{\partial \lambda} \right]'$. For $\lambda > 1$ we have $r_1 < r_2$. For λ close enough to 1, we have either situation where the innovation in either sector is less than 0.5 ($0.5 < \Phi_1 < \Phi_2$, in which case $\varphi_1 > \varphi_2$), or more than 0.5 ($0.5 > \Phi_2 > \Phi_1$, in which case $\varphi_2 > \varphi_1$). In the latter case, we can found $\underline{\varphi} > 0$, such that

$$\frac{\partial^2 u}{\partial r_1 \partial \lambda} + \frac{\partial^2 u}{\partial r_2 \partial \lambda} < -\beta l^* \underline{\varphi}(1 - \Phi_1 - \Phi_2) \frac{\partial \chi^{11} - \chi^{10} - \chi^{01} + \chi^{00}}{\partial \lambda},$$

because using lemma 3 we know that for $\lambda > 1$ we have $\frac{\partial \chi^{11} - \chi^{10} - \chi^{01} + \chi^{00}}{\partial \lambda} < 0$ and hence the sign of $\frac{\partial^2 u}{\partial r_1 \partial \lambda} + \frac{\partial^2 u}{\partial r_2 \partial \lambda}$ depends on the likelihood of success. In particular, if $1 > \Phi_1 + \Phi_2$, we have $\frac{\partial^2 u}{\partial r_1 \partial \lambda} + \frac{\partial^2 u}{\partial r_2 \partial \lambda} < 0$. □

Theorem 6. *If $\lambda > 1$, $\theta \in (0, 1)$ and $1 < \Phi_1 + \Phi_2$, we have*

$$\frac{\partial [r_1^*(\theta, \lambda) + r_2^*(\theta, \lambda)]}{\partial \lambda} > 0.$$

Proof. We want to show that $u_{r_1,\lambda}(u_{r_2,r_2} - u_{r_1,r_2}) + u_{r_2,\lambda}(u_{r_1,r_1} - u_{r_1,r_2}) < 0$. By simple manipulation:

$$u_{r_1,\lambda}(u_{r_2,r_2} - u_{r_1,r_2}) + u_{r_2,\lambda}(u_{r_1,r_1} - u_{r_1,r_2}) = (u_{r_1,\lambda} + u_{r_2,\lambda})(u_{r_2,r_2} - u_{r_1,r_2}) + u_{r_2,\lambda}(u_{r_1,r_1} - u_{r_2,r_2}).$$

First consider the second term. For $\lambda > 1$ we have $r_1 < r_2$ and hence $u_{r_1,r_1} < u_{r_2,r_2} < 0$ which implies $u_{r_1,r_1} - u_{r_2,r_2} < 0$. Furthermore, by lemma 2 we know that $u_{r_2,\lambda} > 0$. Therefore $u_{r_2,\lambda}(u_{r_1,r_1} - u_{r_2,r_2}) < 0$.

Second, by previous consideration $u_{r_2,r_2} - u_{r_1,r_1} < 0$. For proof to be completed it is sufficient that $u_{r_1,\lambda} + u_{r_2,\lambda} < 0$, which is the case for $1 < \Phi_1 + \Phi_2$. □

While the assumption of $1 < \Phi_1 + \Phi_2$ allows this proof to work, numerical investigatino shows that even for $1 > \Phi_1 + \Phi_2$ the result still hods.

E Solution procedure details

E.1 Projection algorithm

I solve the model starting with a guess for the policy functions for production labour $l_t^P = l^P(\mathbf{\Sigma}_t, \boldsymbol{\theta}_{l^P})$ and research labour $l_t^R = l^R(\mathbf{\Sigma}_t, \boldsymbol{\theta}_{l^R})$, where

$$\mathbf{\Sigma}_t = (k_t, \tilde{a}_{1t}, \dots, \tilde{a}_{Nt}, \tilde{e}_{1t}, \dots, \tilde{e}_{Nt}, \mu_{1t}, \dots, \mu_{Nt})$$

is a vector containing all state variables and $\boldsymbol{\theta}_{l^P}$ and $\boldsymbol{\theta}_{l^R}$ are vectors containing the coefficients parameterising the approximation. I use Hermite polynomials as basis functions.

The expectations operator in the Euler equation is evaluated over the distribution of $2N$ independent shocks $\boldsymbol{\mu}_{it+1}, \boldsymbol{\varepsilon}_t$ using Gauss-Hermite quadrature adapted to efficiently account for the kinks in the functions.

1. update $l^P(\Sigma, \boldsymbol{\theta}_{l^P})$:
   - (i) get the wage rage $w = w(\mathbf{\Sigma})$ by combining the condition of the optimal factor input ratio
   - (ii) given $w(\mathbf{\Sigma})$, $l^P(\mathbf{\Sigma}, \boldsymbol{\theta}_{l^P})$ and $l^R(\mathbf{\Sigma}, \boldsymbol{\theta}_{l^R})$, $c(\mathbf{\Sigma})$ can be solved from labour supply equation
   - (iii) k' follows from the budget constraint
   - (iv) multiply the both sides of Euler equation by l to get

     $$l^P = \beta \mathsf{E}\Bigg[\left(g(\mathbf{\Sigma}_{t+1}) \frac{c(\mathbf{\Sigma}_{t+1})}{c(\mathbf{\Sigma}_t)} \right)^{-\gamma} \left(\frac{1 - \left(l^P(\mathbf{\Sigma}_{t+1}, \boldsymbol{\theta}_{l^P}) + l^R(\mathbf{\Sigma}_{t+1}, \boldsymbol{\theta}_{l^R}) \right)}{1 - \left(l^P(\mathbf{\Sigma}_t, \boldsymbol{\theta}_l) + l^R(\mathbf{\Sigma}, \boldsymbol{\theta}_{\tilde{l}}) \right)} \right)^{-\phi} \cdots \\ \times (r(\mathbf{\Sigma}_{t+1}) + 1 - \delta) l^P(\mathbf{\Sigma}_{t+1}, \boldsymbol{\theta}_{l^P})) \Bigg]$$

     and evaluate the right hand side by quadrature on a grid for $\mathbf{\Sigma}$. The grid is constructed following Judd *et al.* (2012): start with a set of simulated data from the model, cluster the points and then choose a subset such that every point the the simulated data is closer than some δ.
   - (v) use these values as a dependent variable and obtain θ_l^{update} by linear projection
   - (vi) apply dampening $\boldsymbol{\theta}_{l^P}^{new} = \lambda \boldsymbol{\theta}_{l^P}^{update} + (1 - \lambda) \boldsymbol{\theta}_{l^P}$ to ensure convergence
   - (vii) update the policy function: $l^P(\mathbf{\Sigma}, \boldsymbol{\theta}_{l^P}) = l^P(\mathbf{\Sigma}, \boldsymbol{\theta}_{l^P}^{new})$

2. update $l^R(\Sigma, \boldsymbol{\theta}_{l^R})$:
   (i) given $l^P(\Sigma, \boldsymbol{\theta}_{l^P})$, iterate or directly solve for $\hat{q}^P(\boldsymbol{\theta}_{l^P})$

   $$q_i^P(\Sigma_t) = \mathsf{E}\left[\bar{\mathbb{1}}_i(\Sigma_t)m(\Sigma_{t+1})g(\Sigma_{t+1})\left\{\pi_i(\Sigma_{t+1}) + q_i^P(\Sigma_{t+1})\right\}\right]$$

   $l^P(\Sigma, \boldsymbol{\theta}_{l^P})$ affects the stochastic discount factor of future profits

   (ii) • given $\hat{q}^P(\boldsymbol{\theta}_{l^P})$, solve for l^R which maximises the value generated in each sector:

   $$l^R = \arg\max \mathsf{E}\left[\mathbb{1}_i(\Sigma_t)m(\Sigma_{t+1})g(\Sigma_{t+1})\left\{\pi_i(\Sigma_{t+1}) + q_i^P(\Sigma_{t+1})\right\}\right] - \tilde{l}(\Sigma_t)w(\Sigma_t)$$

   note that l^R affects the expectations of tomorrows q^P directly by affecting corresponding $\tilde{e}_i$ in the state vector

   • use l^R to update coefficiens $\boldsymbol{\theta}_{l^R}$ a policy rule $l^R(\Sigma, \boldsymbol{\theta}_{l^R})$

3. repeat steps 1 and 2 with dampening factors until $\boldsymbol{\theta}_{l^P}$ and $\boldsymbol{\theta}_{l^R}$ converge

E.2 Construction of grids

Judd *et al.* (2012) propose a grid construction method which ameliorates the curse of dimensionality:

1. normalise the simulated data by subtracting means and dividing by standard deviation.
2. compute the distance matrix
3. cut the outliers (points in the simulation where the number of other points in δ neighborhood is less than some threshold)
4. transform the resulting grid back by inverting the normalisation done in the first step

Unfortunately, by its own design, the projection solution algorithm with ergodic grid is particularly prone to encounter problems due to the bad properties of polynomial approximation away from the area the approximation was constructed over. The reason is that updating the policy function parameters leads to changes in behavior which take the model to different areas of the state space in each iteration. This might mean that the updated policy function takes the model somewhere where there is no guarantee that the approximated solution is still reasonable. For example, assume that the initial guess for the policy function governing labour is such that the agents work less than in the true solution. This means that they accumulate less capital. The grid is hence constructed over a region of state space with low levels of capital. The solution step then updates the policy function so the agents work more. Working more is associated with higher capital accumulation and the policy function which was approximated over the region with lower capital might not work well, especially for higher order polynomial policy function approximations.

There are two ways how to address this issue. First, by smoothing the updating step

$$\theta^{new} = \lambda\theta^{crude} + (1-\lambda)\theta^{old},$$

the extent of movement is reduced at cost of number of iterations needed for convergence.

A second possibility is to stretch the grid to cover a larger area in the first place. However, it is crucial to retain the correlation found in the data for the ergodic grid to retain its properties.

1. let $\mathbf{\Sigma}$ denote the matrix of states from the simulation
2. find a mean in every direction, denote this point by $\bar{\Sigma}$
3. construct stretched matrix of state (with slight abuse of matrix notation): $\mathbf{\Sigma}' = \gamma(\mathbf{\Sigma} - \bar{\Sigma})$
4. construct the grid using $\begin{bmatrix} \mathbf{\Sigma}' \\ \mathbf{\Sigma} \end{bmatrix}$ instead of only $\mathbf{\Sigma}$

The parameter γ has to be chosen in a way that "respects" the simulated data. It is obvious, for example, that capital should not be stretched below zero. However, the model might contain more delicate relationships among variables which need to be respected. For example, in situations where there are variables in the state space linked by a nonlinear relationship, this relationship would be broken by linear stretching suggested in step 3.[24] In such situations, it might still be possible to stretch only one dimension linearly and then re-compute the other dimensions using the nonlinear relationship. It is also possible to stretch the state space only in specific dimensions which are known to be problematic or to simply stretch some variables and avoid the ones with non-linear relationships among specific variables.

Different blocks of the model might also require a different number of points in the grid. In such situations, in might be useful to generate separate grids with different sizes to be used to solve different blocks. The same applies to different stretching.

[24] In Rozsypal (2014a) the relative productivities are computed as a non-linear weighted average of the productivities in different sectors.

F Additional correlation graphs

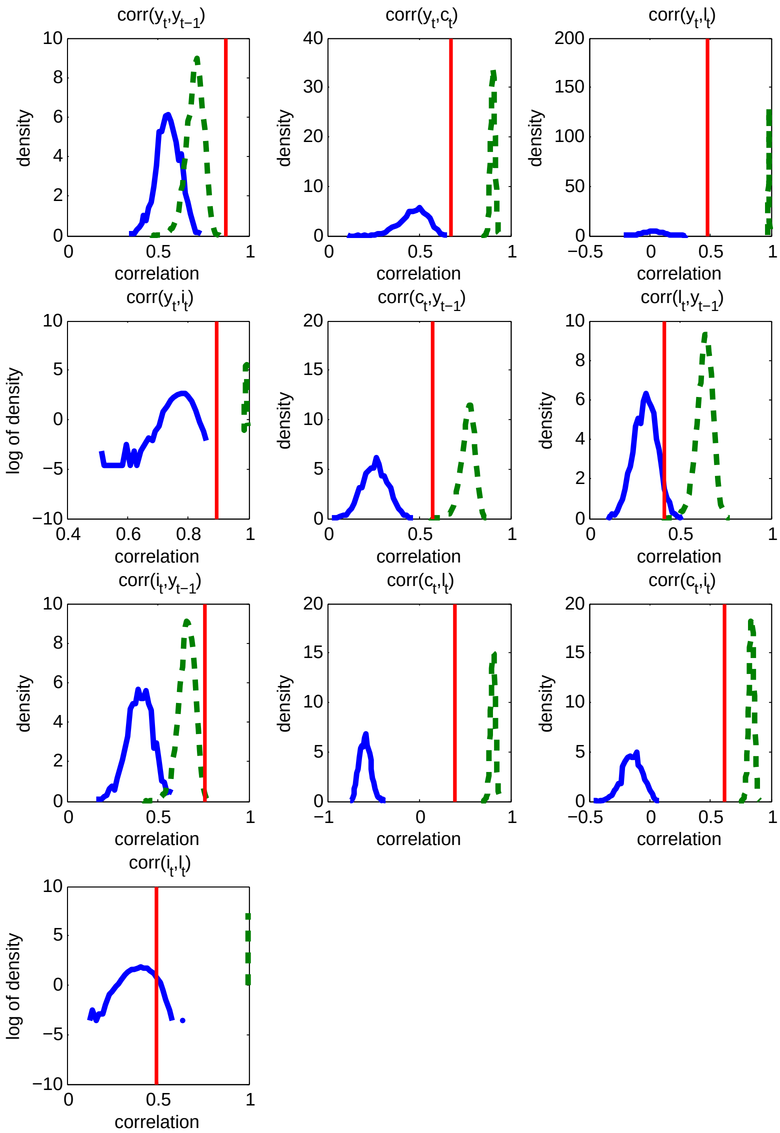


Figure 9: Histogram of correlations (2500 simulations of length 250 periods). The model represented by the solid blue lines, the dashed lines represent a simple RBC model for comparison. The vertical line represents the corresponding correlation from the US data. The densities are normalised so histograms are comparable.

G Generalised impulse-response function

Due to the non-linearity of the discussed class of models, standard impulse response functions (IRF) cannot be employed here. The reason is that a global solution to a nonlinear model allows for different reaction to the same shock depending on the area of the state space. However, generalised impulse response functions (GIRFs) can be generated by the following algorithm.

1. Simulate M series of shocks $(\boldsymbol{\mu}_t, \boldsymbol{\varepsilon}_t)$ of length T and save them into matrix $\boldsymbol{\Omega}^{control}$. This matrix has dimensions $(2N \times T \times M)$ and an element $\boldsymbol{\Omega}^{control}(k,t)$ is the value of k-th shock at period t in all simulations $1,\ldots,M$.[25]

2. Set the value of the shocks in the impact period in the given sector to its mean.

$$\boldsymbol{\Omega}^{test,\mu}(1, \text{impact period}, m) = \mathsf{E}\mu \qquad \forall m$$

3. The fact that all the policy rules are symmetric across the sectors means that it does not really matter which industry is hit with the shock, so without any loss of generality I always hit sector one. Construct $\boldsymbol{\Omega}^{test,\mu}$ by adding an extra value of shock ϵ to the shock μ at time l in industry 1 :

$$\boldsymbol{\Omega}^{test,\mu}(k,t,m) = \begin{cases} \kappa\sigma_\mu & \text{for} \quad l = t, k = 1 \\ \boldsymbol{\Omega}^{control}(k,t,m) & \text{otherwise} \end{cases}$$

   The burn-in period l is important, because the non-linearity of the model means that the effect of a given shock is different for different states of the world. The burn-in period thus allows the model to reach different areas before measuring the response to an impulse.

4. Simulate the model for the series of shocks $\boldsymbol{\Omega}^{test,\mu}(t,m)$, $\boldsymbol{\Omega}^{test,\varepsilon}(t,m)$ and $\boldsymbol{\Omega}^{control}(t,m)$ and obtain corresponding values of endogenous normalised variables $\boldsymbol{x}^{test,\mu}(t,m)$, $\boldsymbol{x}^{test,\varepsilon}(t,m)$ and $\boldsymbol{x}^{control}(t,m)$.

5. Compute levels where needed (capital, output, wages,...) and keep the original variables where appropriate (interest rate, l, $\tilde{l}$) $\boldsymbol{X}^{test,\mu}(t,m)$, $\boldsymbol{X}^{test,\varepsilon}(t,m)$ and $\boldsymbol{X}^{control}(t,m)$.

[25]The ordering of the shocks is the following: at position j=$1,\ldots,N$, there is μ_j and then for j=$N+1,\ldots,2N$ there is ε_j

6. Percentage generalised impulse response function is then generated by

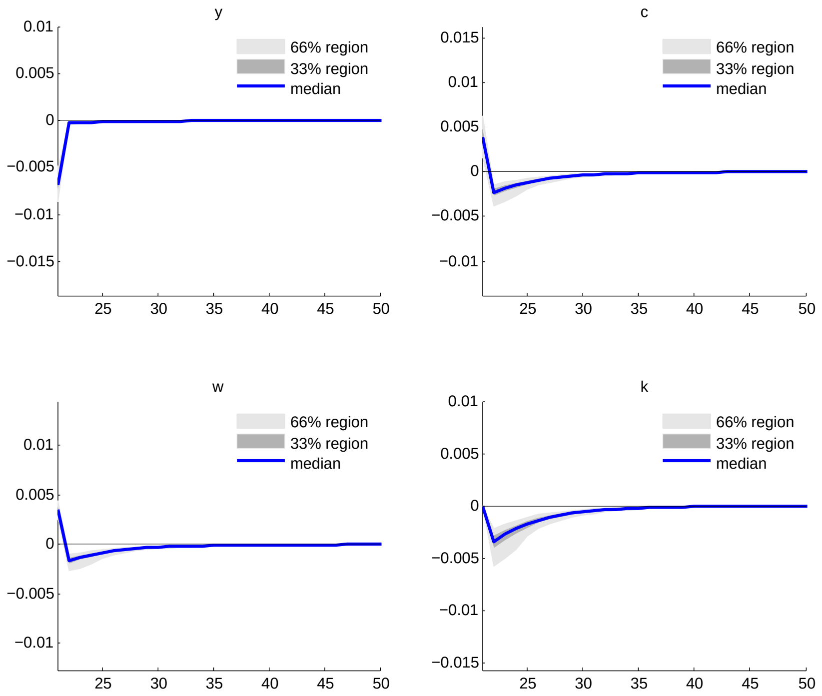
$$\text{girf}^{\mu}(t,m) = \frac{\boldsymbol{X}^{test,\mu}(t,m) - \boldsymbol{X}^{control}(t,m)}{\boldsymbol{X}^{control}(t,m)}.$$

7. Sort the result over dimension m and report percentiles to capture the nonlinearity of the model.

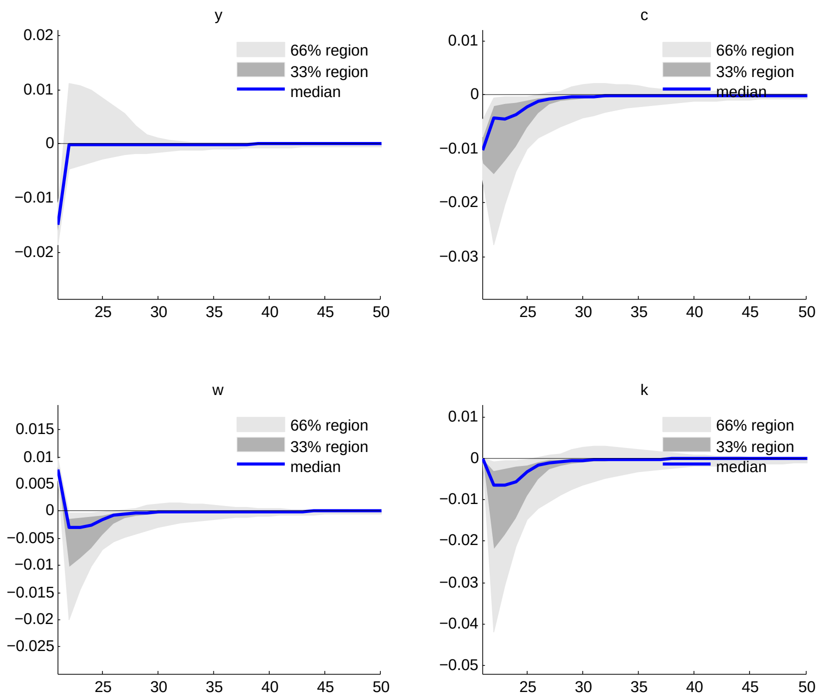
H Additional impulse response functions

Note that a *good* realisation of μ_{it} leads to subsequent *fall* in many variables. Let's explain the logic behind this result on *normalised* variables using capital; good realisation of μ leads to innovation which increases the the level of productivity $\tilde{A}$. Even if the not normalised level of capital K stays the same, the normalised $k = K/\tilde{A}$ falls because of the increase in $\tilde{A}$. Only subsequently the capital start to grow back to the steady state level.

Variables like l (total labour), l^P (labour in production sectors), l^R labour in research sectors or interest rate r, are not normalised because they do not grow along the balanced growth path.

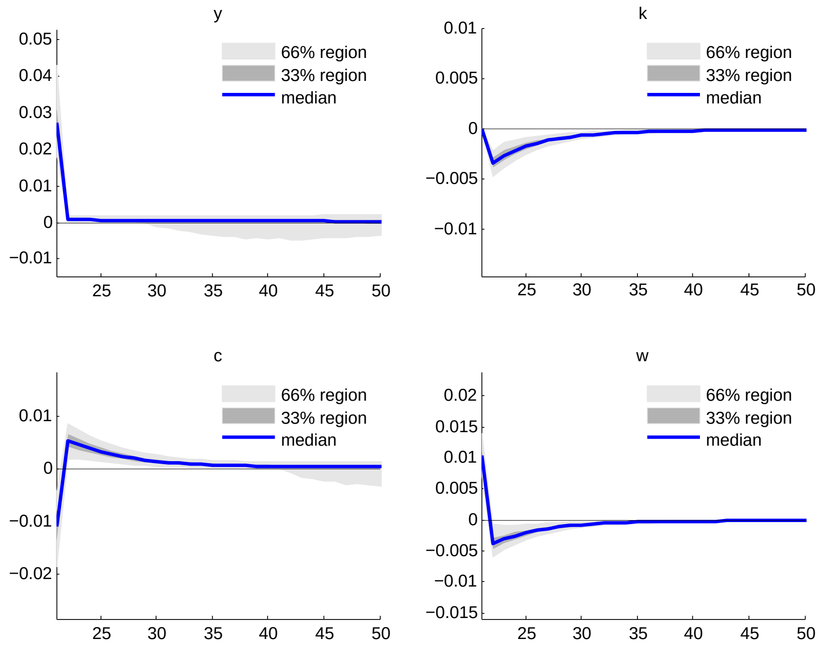


(a) 1 standard deviation shock

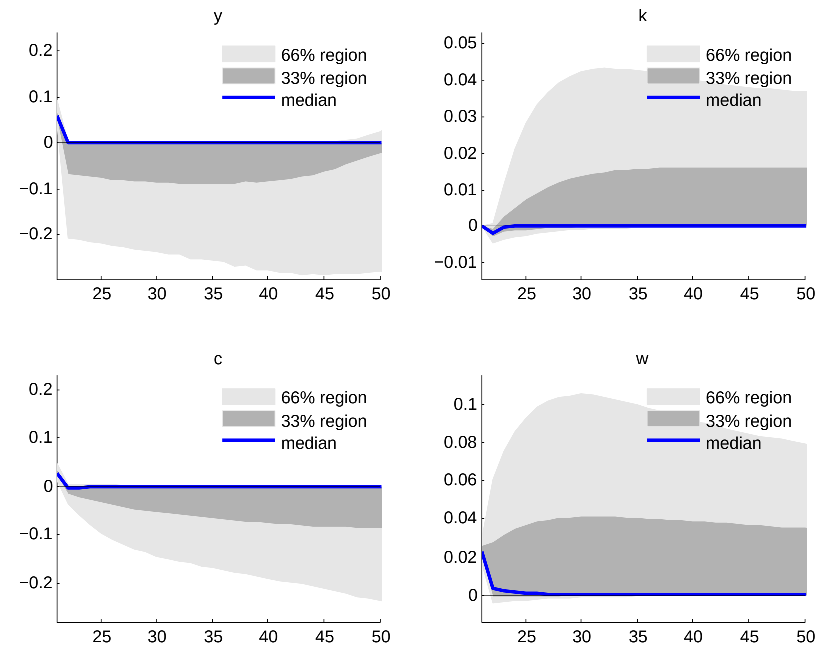


(b) 2 standard deviations shock

Figure 10: IRF on normalised variables



(a) 1 standard deviation shock



(b) 2 standard deviations shock

Figure 11: IRF on level variables

I Basic RBC model

The simple RBC model used for comparison is defined by the following set of equations:

$$
\begin{aligned}
\frac{1}{c_t} &= \beta \mathsf{E}\left[\frac{1}{c_{t+1}}(1+\alpha k_{t+1}^{\alpha-1} e^{z_{t+1}} l_{t+1}^{1-\alpha} - \delta)\right], \\
\psi \frac{c_t}{1-l_t} &= (1-\alpha) k_{t-1}^{\alpha} (e^{z_t})^{1-\alpha} l_t^{-\alpha}, \\
c_t + i_t &= y_t, \\
y_t &= k_{t-1}^{\alpha} (e^{z_t} l_t)^{1-\alpha}, \\
i_t &= k_t - (1-\delta) k, \\
z_t &= \rho z_{t-1} + \varepsilon_t,
\end{aligned}
$$

where $\alpha = 0.33$, $\beta = 0.99$, $\delta = 0.025$, $\psi = 1.75$, $\rho = 0.95$, $\sigma = (0.007/(1-\alpha))$