Socially efficient discounting under ambiguity aversion

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Abstract

We consider an economy with an ambiguity-averse representative agent who faces uncertain consumption growth. We examine conditions under which ambiguity aversion reduces the socially efficient discount rate. It is shown that ambiguity aversion affects the interest rate in two ways. The first effect is an ambiguity prudence effect, similar to the prudence effect that prevails in the expected utility model. In contrast, it requires decreasing ambiguity aversion in order to be signed. The second effect is that ambiguity also entails pessimism. But this implicit shift in beliefs generally has an ambiguous effect on the interest rate. We provide sufficient conditions under which ambiguity aversion does indeed decrease the socially efficient discount rate. The calibration of the model suggests that the effect of ambiguity aversion on the way we should discount distant cash flows is potentially large.

Keywords: Decreasing ambiguity aversion, ambiguity prudence, Ramsey rule, sustainable development.

1 Introduction

The emergence of public policy problems associated with the sustainability of our development has raised considerable interest for the determination of a socially efficient discount rate. This debate has recently culminated in the publication of two reports about the evaluation of different public investments. On one side, the Copenhagen Consensus (Lomborg (2004)) put top priority on public programs yielding immediate benefits (fighting malaria and AIDS, improving water supply,...), and rejected the idea to invest much in the prevention of global warming. On the other side, the Stern Review (Stern (2007)) put tremendous pressure on acting quickly and heavily against global warming.

Because global warming will really affect our economies in a relatively distant time horizon, the choice of the rate at which these costs are discounted plays a key role in reaching either conclusion. While Stern applies an implicit rate of 1.4% per year, the Copenhagen Consensus argues that an efficient rate should be around 5%. For the sake of illustrating the power of discounting, consider a project which yields its benefits in t years time. For a horizon $t = 100$ the Copenhagen Consensus would require a rate-of-return already 36 times higher than Stern.

As stated by the well-known Ramsey rule (Ramsey (1928)), the socially efficient discount rate (net of the rate of pure preference for the present) is equal to the product of relative risk aversion and the growth rate of consumption. The basic idea is that, given the assumption that one will be wealthier in the future, one is willing to improve future wealth by sacrificing current wealth only if the return on this investment is large enough to compensate for the increased intertemporal inequality that it generates. If we assume that the growth rate of wealth is 2% and relative risk aversion equals 2, this yields a discount rate of 4%.

However, if one wants to use this reasoning to value investments affecting distant generations, it is crucial to take into account the riskiness affecting the long-term growth of consumption. Hansen and Singleton (1983), Gollier (2002) and Weitzman (2007a), among others, have extended the Ramsey rule by assuming an exogenously given stochastic growth process. This adds a precautionary term to the Ramsey rule which tends to reduce the discount rate in order to induce more investment for the future. The convexity of the prudent representative agent's marginal utility implies that the uncertainty about future consumption raises the expected marginal utility, i.e. the willingness to save for the future (Leland (1968), Drèze and Modigliani (1972)). This reduces the interest rate.

The present paper goes one step further in recognizing the potential uncertainty on the long-term growth process itself. Such parameter uncertainty on priors is typically referred to as statistical ambiguity or Knightian uncertainty. We believe that this assumption is realistic, especially for long-term forecasts.

Departing from the standard Subjective Expected Utility paradigm (SEU, Savage (1954)), we also assume that the representative agent is ambiguity-averse, i.e., that she dislikes mean-preserving spreads over prior beliefs. Indeed, starting with the pioneering work by Ellsberg (1961), ample evidence in favor of this hypothesis has been accrued.[1] All of which suggests that it is behaviorally meaningful to distinguish lotteries over prior distributions from lotteries over final outcomes. In what follows, we will consider a representative agent who displays "smooth ambiguity preferences", as recently proposed by Klibanoff, Marinacci and Mukerji (KMM, 2005, 2007). Accordingly, the agent computes the expected utility of future consumption conditional on each possible value of the uncertain parameter. She then evaluates her future felicity by computing the certainty equivalent of these conditional expected utilities, using an increasing and concave function ϕ. The concavity of this function implies that she dislikes any mean-preserving spread in the set of plausible beliefs, i.e. that she is ambiguity-averse. Also, it was shown by KMM that the smooth ambiguity family entails the well-known max-min criterion as a special case.

In this paper, we address the question of how ambiguity aversion affects the socially efficient discount rate. Intuitively, we might expect that it should raise the agent's willingness to save in order to compensate for the adverse effect of ambiguity on future welfare. It turns out, however, that this is not true in general: ambiguity aversion may increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utilities. On the one hand, there is an ambiguity prudence effect, similar to the prudence effect in the expected utility framework. We show that the mere uncertainty on the conditional expected utility reduces its ϕ-certainty equivalent if and only if ϕ exhibits decreasing absolute (ambiguity) aversion (DAAA). The reason why merely demanding the convexity of ϕ' is not enough is precisely that future felicity is measured by the ϕ-certainty equivalent rather than by the expectation of ϕ.

On the other hand, as observed by KMM (2005, 2007), ambiguity aversion yields an implicit pessimism effect, which acts as if probability weights were shifted towards more unfavorable prior distributions, in the sense of the Monotone Likelihood Ratio order (MLR). However, this shift in beliefs does not in general imply a reduction of the interest rate. We derive pairs of conditions on the risk attitude and on the stochastic ordering of plausible distributions which guarantee that, under DAAA, the socially efficient discount rate is lower than in the ambiguity-neutral benchmark.

This paper is related to Weitzman (2007a) and Gollier (2007b), who also recognize the uncertainty affecting the growth of the economy as an important

[1] The *Ellsberg-Paradox* refers to the outcome of an experiment (Ellsberg (1961)). In an urn containing 90 balls there were 30 red balls, and the remaining were either black or yellow in unknown proportions. Participants had to bet on the color of the ball drawn, receiving a prize of \$100 in case of a successful bet. A large group displayed the following behavioral pattern: On the one hand they preferred to bet on drawing *red* vs. betting on *black*. However, in a second stage they preferred to bet on *not* drawing *red* vs. betting on *not* drawing *black*. This choice pattern contradicts the hypothesis that participants associate unique subjective probabilities to each outcome of a draw, as required in the SEU framework. Note that betting on (or against) *red* is indeed an unambiguous act with well-defined winning probabilities, while betting on (or against) *black* is not. For a survey of the literature consult e.g. Camerer and Weber (1992).

feature of the discounting problem. Weitzman (2007a) shows that the uncertainty affecting the volatility of the growth process may yield a term structure of the discount rate that tends to minus infinity for very long time horizons. Gollier (2007b) provides a typology of more general structures on the parametric uncertainty. He shows that the sign of the third or fourth derivative of the utility function are necessary to sign the effect on the efficient discount rate, depending upon its type. We depart strongly from these works – all of which are based on the SEU approach – in allowing for ambiguity-sensitive preferences.

Jouini, Napp and Marin (2008) and Gollier (2007a) consider the related question of how to aggregate diverging beliefs in a SEU framework. Jouini, Napp and Marin show that an aggregation bias might cause a richer evolution of the discount rate than in the representative agent models. In particular, the discount rate might be first increasing and only then approach its limit, namely the smallest individual rate.

The most active branch of the literature on ambiguous processes deals with asset pricing. Clearly, the underlying mechanisms are very similar to the ones we will study below. Methodologically, our paper is most closely related to Gollier (2006). He investigates comparative statics results of an increase in ambiguity aversion on the demand for risky assets. It turns out that, in general, omitting ambiguity aversion cannot be corrected for by assuming a higher degree of risk aversion.

More concretely, Ju and Miao (2007) and Collard, Mukerji, Sheppard and Tallon (2008) investigate the evolution of asset prices numerically. Using tractable functional forms, they show that, indeed, several empirical phenomena, like the relatively low risk-free rates, can be matched in a KMM framework. However, as the present paper shows, the negative relation between the degree of ambiguity aversion and the risk-free rate does not hold for more general KMM specifications.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. We decompose the effect of ambiguity aversion into its two components in Section 4, whereas Sections 5 and 6 are devoted to respectively the ambiguity prudence effect and the pessimism effect. Section 7 investigates under which conditions our findings extend to any increase in ambiguity aversion. Finally, before concluding, we calibrate the model using two different specifications in Section 8.

2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\widetilde{c}_t$ fruits at date t, $t = 0, 1, 2, ...$ There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{r_t t}$ fruits for sure at date t. Thus, the real interest rate associated to maturity t is r_t. The distribution of $\widetilde{c}_t$

is a function of a parameter θ that can take values $1, 2, ..., n$. This parametric uncertainty takes the form of a random variable $\widetilde{\theta}$ whose probability distribution is a vector $q = (q_1, ..., q_n)$, where q_θ is the probability that $\widetilde{\theta}$ takes value θ. The cumulative distribution function of $\widetilde{c}_t$ conditional to θ is denoted $F_{t\theta}$. The crop conditional to θ is denoted $\widetilde{c}_{t\theta}$. An ambiguous environment for $\widetilde{c}_t$ is thus fully described by $\widetilde{c}_t \sim (\widetilde{c}_{t1}, q_1, ; ...; \widetilde{c}_{tn}, q_n)$. Conditional to θ, the expected utility of an agent who purchases α zero-coupon bonds with maturity t equals

$$U_t(\alpha, \theta) = Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) = \int u(c + \alpha e^{r_t t}) dF_{t\theta}(c).$$

We assume that u is three times differentiable, increasing and concave, so that $U(., \theta)$ is concave in the investment α, for all θ.

Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji (2007)), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. Ex ante, for a given investment α, her welfare is measured by $V_t(\alpha)$, which is the certainty equivalent of the conditional expected utilities:

$$\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(U_t(\alpha, \theta)) = \sum_{\theta=1}^{n} q_\theta \phi\left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})\right). \tag{1}$$

Function ϕ describes the investor's attitude towards ambiguity (or parameter uncertainty). It is assumed to be three times differentiable, increasing and concave. A linear function ϕ means that the investor is neutral to ambiguity. In such a case, the decision maker is indifferent to any mean-preserving spread of $U_t(\alpha, \widetilde{\theta})$. Thus her preferences can be represented by a subjective expected utility functional $V_t^{SEU}(\alpha) = Eu(\widetilde{c}_t + \alpha e^{r_t t})$. On the contrary, a concave ϕ is synonymous of ambiguity aversion in the sense that one dislikes any mean-preserving spread of the conditional expected utility $U_t(\alpha, \widetilde{\theta})$. An interesting particular case arises when absolute ambiguity aversion $A(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -A^{-1}\exp(-AU)$. As proven by Klibanoff, Marinacci and Mukerji (2005), the ex-ante welfare $V_t(\alpha)$ tends to the max-min expected utility functional $V_t^{MEU}(\alpha) = \min_\theta Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t})$ when the degree of absolute ambiguity aversion ϕ tends to infinity. Thus, the max-min criterion à la Gilboa and Schmeidler (1989) is a special case of this model.

The optimal investment α^* maximizes the intertemporal welfare of the investor, which is written as

$$\alpha^* \in \arg\max_{\alpha} \qquad u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha). \tag{2}$$

where parameter δ is the rate of pure preference for the present.

At this stage, it is important to point out that the basic assumptions underlying KMM models do not guarantee that the maximization problem (2) is convex. To see why, it suffices to recall that certainty equivalent functions need not be concave. Indeed, even if we imposed ϕ and u to be strictly concave, the solution to program (2), when it exists, need not be unique. However, we can prove the following.

Proposition 1 *Suppose that ϕ has a concave absolute ambiguity tolerance, i.e., $-\phi'(U)/\phi''(U)$ is concave in U. This implies that V_t is concave in α.*

Proof. Relegated to the Appendix.

If the inverse of absolute ambiguity aversion increases at a linear or decreasing rate in U, then the KMM functional is concave in α. The above proposition includes the specifications which are most widely used in the literature: most importantly the family of exponential functions and the family of power functions.

Henceforward we will consider the following assumption satisfied.

Assumption 1 *The function ϕ exhibits a concave absolute ambiguity tolerance, i.e., $-\phi'(U)/\phi''(U)$ is concave in U everywhere.*

Thanks to Assumption 1, the necessary and sufficient condition to solve program (2) can be written as

$$u'(c_0 - \alpha^*) = e^{-\delta t} V_t'(\alpha^*).$$

Fully differentiating equation (1) with respect to α yields

$$V_t'(\alpha) = e^{r_t t} \frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta} + \alpha e^{r_t t}) \right) Eu'(\widetilde{c}_{t\theta} + \alpha e^{r_t t})}{\phi'(V_t(\alpha))}.$$

Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity t is $\alpha^* = 0$. Combining the above two equations implies the following equilibrium condition:

$$r_t = \delta - \frac{1}{t} \ln \left[\frac{\sum_{\theta=1}^{n} q_\theta \phi' \left(Eu(\widetilde{c}_{t\theta}) \right) Eu'(\widetilde{c}_{t\theta})}{\phi'(V_t(0)) u'(c_0)} \right]. \tag{3}$$

This is also the socially efficient rate at which sure benefits and costs occurring at date t must be discounted in any cost-benefit analysis at date 0.

As a benchmark, consider an ambiguity neutral representative agent. In this case we retrieve the standard bond pricing formula $r_t = \delta - t^{-1} \ln \left[Eu'(\widetilde{c}_t)/u'(c_0) \right]$.[2] In this special case, we see that the riskiness of future consumption reduces the socially efficient discount rate if and only if $Eu'(\widetilde{c}_t)$ is larger than $u'(E\widetilde{c}_t)$, i.e., if and only if u' is convex, or if the representative agent is *prudent* (Leland (1968), Drèze and Modigliani (1972), Kimball (1990)).

Our goal in this paper is to determine the conditions under which ambiguity aversion reduces the discount rate. An ambiguous environment $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$ is said to be acceptable if the respective supports of the $\widetilde{c}_{t\theta}$ are in the domain of u, and if all $Eu'(\widetilde{c}_{t\theta})$ are in the domain of ϕ. The set of acceptable ambiguous environments is denoted Ψ.

[2] See for example Cochrane (2001).

3 An analytical solution

Let us consider the following specification:

- The plausible distributions of $\ln \widetilde{c}_{t\theta}$ are all normal with the same variance $\sigma^2 t$, and with mean $\ln c_0 + \theta t$.[3]
- The parameter θ is normally distributed with mean μ and variance σ_0^2.[4]
- The representative agent's preferences exhibit constant relative risk aversion $\gamma = -cu''(c)/u'(c)$, i.e., $u(c) = c^{1-\gamma}/(1-\gamma)$.
- The representative agent's preferences exhibit constant relative ambiguity aversion $\eta = -|u|\,\phi''(u)/\phi'(u) \geq 0$. This means that $\phi(U) = k(kU)^{1-\eta k}/(1-\eta k)$, where $k = sign(1-\gamma)$ is the sign of u.

As is well-known, the Arrow-Pratt approximation is exact under CRRA and lognormally distributed consumption. Therefore, conditional to each θ, we have that

$$Eu(\widetilde{c}_{t\theta}) = (1-\gamma)^{-1}\exp(1-\gamma)(\ln c_0 + \theta t + 0.5(1-\gamma)\sigma^2 t).$$

We can again use the same trick to compute the ϕ-certainty equivalent V_t, since $\phi(Eu(\widetilde{c}_{t\theta}))$ is an exponential function and the random variable $\widetilde{\theta}$ is normal, which is another case where the Arrow-Pratt approximation is exact. It yields

$$V_t(0) = (1-\gamma)^{-1}\exp(1-\gamma)\Big(\ln c_0 + \mu t + 0.5(1-\gamma)\sigma^2 t + 0.5(1-\gamma)(1-k\eta)\sigma_0^2 t^2\Big).$$

However, in order to solve for the pricing rule (3) we are really interested in $V_t'(0)$. A convenient way to structure the algebra is to decompose $V_t'(0)$ in the following way: again exploiting the Arrow-Pratt approximation, we have on the one hand

$$\frac{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)}{\phi'(V_t(0))} = \exp\Big(\frac{1}{2}(1-\gamma)^2 k\eta\sigma_0^2 t^2\Big), \tag{4}$$

and on the other hand

$$\begin{aligned}\frac{E[\phi'\left(Eu(\widetilde{c}_{t\theta})\right)Eu'(\widetilde{c}_{t\theta})]}{E\phi'\left(Eu(\widetilde{c}_{t\theta})\right)} = \quad & \exp-\Big(\gamma(\ln c_0 + \mu t) \quad - \frac{1}{2}\gamma^2(\sigma^2 t + \sigma_0^2 t^2) - \\ & -(\gamma(1-\gamma)k\eta)\sigma_0^2 t^2\Big). \end{aligned} \tag{5}$$

Finally, multiplying expressions (4) and (5) and plugging the result into (3), yields the desired analytical expression:

$$r_t = \delta + \gamma\mu - \frac{1}{2}\gamma^2(\sigma^2 + \sigma_0^2 t) - \frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2 t. \tag{6}$$

[3] In continuous time, this would mean that the consumption process is a geometric brownian motion $d\ln c_t = \theta dt + \sigma dw$.

[4] We consider the natural continuous extension of our model with a discrete distribution for $\widetilde{\theta}$.

Let us define g as the expected growth rate of consumption. It is easy to check that $g = \mu + 0.5(\sigma^2 + \sigma_0^2 t)$. It implies that the above equation can be rewritten as

$$r_t = \delta + \gamma g - \frac{1}{2}\gamma(\gamma+1)(\sigma^2 + \sigma_0^2 t) - \frac{1}{2}\eta \left|1-\gamma^2\right| \sigma_0^2 t. \tag{7}$$

The first two terms on the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the expected growth rate of consumption g. When g is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\widetilde{c}_t)$ under prudence, this has a negative impact on the discount rate.[5] Notice that the variance of consumption at date t equals $\sigma^2 t + \sigma_0^2 t^2$, so that it increases at an increasing rate with respect to the time horizon. Therefore, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2007b) to justify a decreasing discount rate in an expected utility framework.

The last term in the right–hand side of equation (7) characterizes the effect of ambiguity. Observe that it always tends to reduce the discount rate under positive ambiguity aversion ($\eta > 0$). This effect is increasing in the degree of ambiguity aversion η, in the degree of uncertainty σ_0, and in the time horizon t. This implies that more effort will be exerted to improve the ambiguous future.

Observe, that in our example, in the absence of ambiguity (i.e. $\sigma_0^2 = 0$), the term structure is flat. The mere presence of ambiguity (i.e. $\sigma_0^2 > 0$ but $\eta = 0$) causes the rates to decrease linearly over time. Introducing ambiguity aversion steepens this decline.

The following sections investigate whether it is true in general, that ambiguity aversion decreases the socially efficient discount rate for any maturity. Contrary to the example presented above, the next section reveals that ambiguity aversion might even decrease the willingness to save.

4 The two effects of ambiguity aversion

Consider first the benchmark case of an ambiguity-neutral representative agent, where the discount rate equals

$$r_t = \delta - \frac{1}{t}\ln\left[\frac{Eu'(\widetilde{c}_t)}{u'(c_0)}\right]. \tag{8}$$

The random variable $\widetilde{c}_t$ describes future consumption, which is distributed as $(\widetilde{c}_{t1}, q_1; ...; \widetilde{c}_{tn}, q_n)$.

[5] This precautionary effect is equivalent to reducing the growth rate of consumption g by the precautionary premium (Kimball (1990)) $0.5(\gamma+1)(\sigma^2+\sigma_0^2 t)$. Indeed, $\gamma+1 = -cu'''(c)/u''(c)$ is the index of relative prudence of the representative agent.

Just like in the analytical example above, we can decompose $V_t'(0)$ such that the pricing rule under ambiguity aversion can be written as

$$r_t = \delta - \frac{1}{t}\ln\left[a\frac{Eu'(\overset{\circ}{\widetilde{c}_t})}{u'(c_0)}\right], \tag{9}$$

where the constant a is defined as

$$a = \frac{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)}{\phi'(V_t(0))}, \tag{10}$$

and where $\overset{\circ}{\widetilde{c}_t}$ is a distorted probability distribution $(\widetilde{c}_{t1}, \overset{\circ}{q}_1; ...; \widetilde{c}_{tn}, \overset{\circ}{q}_n)$ of future consumption, with

$$\overset{\circ}{q}_\theta = \frac{q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)}{\sum_{\tau=1}^{n} q_\tau \phi'\left(Eu(\widetilde{c}_{t\tau})\right)}, \tag{11}$$

for $\theta = 1, ..., n$.

Notice the similarity between pricing formula (3) and the benchmark (8). It implies that ambiguity aversion reduces the discount rate if

$$aEu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t). \tag{12}$$

Moreover, Observe that this condition simplifies to $a \geq 1$ when the agent is risk neutral. Because we don't constrain the risk attitude in any way except risk aversion, condition $a \geq 1$ is necessary to guarantee that ambiguity aversion reduces the discount rate. For reasons that will be clarified in the next section, we will refer to $a \geq 1$ as the ambiguity prudence effect.

In the absence of an ambiguity prudence effect ($a = 1$), condition (12) becomes $Eu'(\overset{\circ}{\widetilde{c}_t}) \geq Eu'(\widetilde{c}_t)$, which is referred to as the pessimism effect. At this stage, it is enough to say that it comes from a distortion of the beliefs $(q_1, ..., q_n)$ on the likelihood of the different plausible probability distributions $(\widetilde{c}_1, ..., \widetilde{c}_n)$.

5 The ambiguity prudence effect

In this section, we focus on whether the constant a, defined by equation (10), is larger than unity. As stated above, this is necessary to guarantee that the discount rate is reduced and it becomes necessary and sufficient in the special case of risk-neutrality. Notice that in the latter case, a can be interpreted as the sensitiveness of the $\phi-$certainty equivalent of $\bar{c}_{\widetilde{\theta}} = E\left[\widetilde{c}_{t\widetilde{\theta}} \mid \widetilde{\theta}\right]$ with respect to an increase in saving.[6] The problem is thus to determine whether one more dollar saved yields an increase in the $\phi-$certainty equivalent future consumption. More generally, condition $a \geq 1$ can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \phi'\left(u_\theta\right) \geq \phi'(V_t) \text{ whenever } \Sigma_\theta q_\theta \phi(u_\theta) = \phi(V_t). \tag{13}$$

[6] Define $V(s, \bar{c}_{\widetilde{\theta}})$ such that $\phi(s+V) = E\phi(s+\bar{c}_{\widetilde{\theta}})$. We have that $a = \partial V(s, \bar{c}_{\widetilde{\theta}})/\partial s$ at $s = 0$.

In words, do expected-utility-preserving risks raise expected marginal utility, where the utility function referred here is the ϕ function? The answer to this question is well-known in expected utility theory (see e.g. Gollier (2001, section 2.5)). This is true if and only if ϕ exhibits decreasing absolute ambiguity aversion. Indeed, defining function ψ such that $\psi(\phi(U)) = \phi'(U)$ for all U, the above condition can be rewritten as

$$\sum_{\theta=1}^{n} q_\theta \psi\left(\phi_\theta\right) \geq \psi(\Sigma_\theta q_\theta \phi_\theta),$$

where $\phi_\theta = \phi(u_\theta)$ for all θ. This is true for all distributions of $(\phi_1, q_1; ...; \phi_n, q_n)$ if and only if ψ is convex. Because $\psi'(\phi(U)) = \phi''(U)/\phi'(U)$, this is true iff $A(U) = -\phi''(U)/\phi'(U)$, which is the index of absolute ambiguity aversion, be non-increasing. This proves the following results.

Lemma 1 *$a \geq 1$ (resp. $a \leq 1$) for all acceptable ambiguous environments $\widetilde{c} \in \Psi$ if and only if absolute ambiguity aversion is non-increasing (resp. non-decreasing).*

Proposition 2 *Suppose that the representative agent is risk-neutral. The socially efficient discount rate is smaller (resp. larger) than under ambiguity neutrality for all ambiguous environments $\widetilde{c}$ if and only if ϕ exhibits non increasing (resp. non decreasing) absolute ambiguity aversion.*

Under risk neutrality, the driving force for the impact of ambiguity on the interest rate is not ambiguity aversion itself, but rather whether the degree of ambiguity aversion is increasing or decreasing with the level of conditional expected utility U. In the limit case, with risk neutrality and constant absolute ambiguity aversion, ambiguity has *no* effect on the equilibrium interest rate. The intuition for these results is easy to derive from the observation that the period-t felicity V_t is approximately equal to expected consumption minus the ambiguity premium. Moreover, the premium is itself proportional to ambiguity aversion A, which makes the willingness to save decreasing in A'. Thus, ambiguity aversion raises the willingness to save – therefore reducing the equilibrium interest rate – if absolute ambiguity aversion is decreasing.

Exactly as decreasing absolute risk aversion is unanimously accepted as a natural assumption for risk preferences, we believe that decreasing absolute ambiguity aversion (DAAA) is a reasonable property of uncertainty preferences. It means that a local mean-preserving spread in conditional expected utility has an impact on welfare that is decreasing in the level of utility where this spread is realized.

We call this the *ambiguity prudence effect* because it emerges as a consequence of the uncertainty of the future conditional expected utility. This raises the willingness to save exactly as the risk on future income raises savings in the standard expected utility model under "risk prudence". But contrary to risk prudence, which is characterized by $u''' \geq 0$, ambiguity prudence is described by decreasing absolute uncertainty aversion, which is weaker than $\phi''' \geq 0$. This is because, in the intertemporal KMM model, the future felicity is represented by the $\phi-$certainty equivalent of the conditional expected utilities, rather than by the expected $\phi-$valuation of the conditional expected utilities. If we would have used this alternative model, ϕ' convex would have been the necessary and sufficient condition to sign the ambiguity prudence effect.

However, once we allow for risk aversion, another effect emerges, and non increasing ambiguity aversion is not sufficient anymore to unambiguously sign the effect of ambiguity on the discount rate. This is shown by the following counter-example.

> **Counter-example 1.** Let c_0 equal 2. We assume that $\widetilde{c}_t$ has two plausible distributions, $\widetilde{c}_{t1} \sim (1, 1/3; 4, 1/3; 7, 1/3)$ and $\widetilde{c}_{t2} \sim (3, 2/3; 4, 1/3)$. We assume that these two distributions are equally likely to be the true one, i.e., $q_1 = q_2 = 1/2$. We assume that the agent exhibits constant relative risk aversion (CRRA) with $\gamma = 2$, i.e., $u(c) = -c^{-1}$. We assume that the rate of pure preference for the present δ equals zero. It is easy to check that the interest rate equals 9.24% in that economy if the representative agent would be neutral to ambiguity. Suppose alternatively that she has constant absolute ambiguity aversion (CAAA) with $A = 2.11$, i.e., $\phi(U) = -\exp(-2.11U)$. Then, tedious computations lead to the conclusion that the socially efficient discount rate should be exactly zero: $r_t = 0$! Thus, this example demonstrates that DAAA is not enough to guarantee that ambiguity about future consumption reduces the discount rate.$\blacksquare$

6 The pessimism effect

Counter-example 1 can be explained by the presence of a second effect, the pessimism effect. In the pricing formula (9), the expected marginal utility is computed, using the distorted random variable $\widetilde{c}_t^{\circ}$ rather than the original $\widetilde{c}_t$. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (11). This section is devoted to characterize how the distortion affects the discount rate. If we find that it is pessimistic in the sense of FSD, then we are able to unambiguously sign the effect on the discount rate.

To examine this specific question, we begin by comparing of the distorted probabilities $q^{\circ} = (q_1^{\circ}, ..., q_n^{\circ})$ to the original probabilities $q = (q_1, ..., q_n)$.

Suppose that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$, i.e. that priors are ranked in such a way that the agent always prefers a larger θ. We hereafter show that ambiguity aversion is equivalent to a distortion of the prior beliefs on parameter $\widetilde{\theta}$ in the sense of the Monotone Likelihood Ratio Order (MLR). By definition, a shift of beliefs from q to q° entails a deterioration in the sense of the *monotone likelihood ratio ordering* (MLR) if $q_{\widetilde{\theta}}^{\circ}/q_{\widetilde{\theta}}$ and $\tilde{\theta}$ are anti-comonotonic. Observe from (11) that $q_{\theta}^{\circ}/q_{\theta}$ is proportional to $\phi'(Eu(\widetilde{c}_{t\theta}))$. Thus, since ϕ' is decreasing, we know that $q^{\circ}_{\widetilde{\theta}}/q_{\widetilde{\theta}}$ and $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ are anti-comonotonic. By transitivity, we can state the following.

Lemma 2 *The subsequent conditions are equivalent:*

1. *Beliefs q° are dominated by q in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq Eu(\widetilde{c}_{t2}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*
2. *ϕ is concave.*

This is a consequence of a well-known result on stochastic orderings (see Lehmann (1955)). Another economic application can be found in Quiggin (1995), who studies probability transformations in rank dependent utility models.

The intuitive interpretation is that ambiguity aversion is characterized by an MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\widetilde{c}_{t\theta}$ to marginal $\widetilde{c}_{t\theta'}$, then, the ambiguity-averse representative agent increases the implicit prior probability q°_{θ} relatively more than the implicit prior probability $q^{\circ}_{\theta'}$. This result gives some flesh to our pessimism terminology. It also generalizes – and builds a bridge to – the maxmin case where all the weight is transferred to the worst θ.

Intuitively, this worsening of the future risk should induce the representative consumer to raise her saving. However, the MLR deterioration in the distribution $\widetilde{\theta}$ of the priors is not enough to ensure a negative pessimism effect. Indeed, this is exactly what we observe in counterexample 1. Instead, the crucial requirement would be that probability distortion raises the unconditional expected marginal utility. That is, overweighing scenarios which yield larger conditional expected marginal utility. The above lemma says something different: it states that the probability distortion overweighs scenarios which yield lower expectations on utility. To solve this problem, we need that the conditional Eu and Eu' be ranked in opposite directions.

Lemma 3 *The following two conditions are equivalent:*

1. *The pessimism effect reduces the discount rate, i.e. $Eu'(\widetilde{c}^{\circ}_t) \geq Eu'(\widetilde{c}_t)$, for all ϕ increasing and concave;*
2. *$E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic.*

Proof: To prove that $2 \Rightarrow 1$, suppose that $E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right]$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ be anti-comonotonic. Since ϕ' is decreasing, our assumption implies that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are comonotonic. By the covariance rule, it implies that

$$\begin{aligned} Eu'(\overset{\circ}{\widetilde{c}_t}) &= \frac{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right) Eu'(\widetilde{c}_{t\theta})}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\ &\geq \frac{\left[\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)\right]\left[\sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta})\right]}{\sum_{\theta=1}^{n} q_\theta \phi'\left(Eu(\widetilde{c}_{t\theta})\right)} \\ &= \sum_{\theta=1}^{n} q_\theta Eu'(\widetilde{c}_{t\theta}) = Eu'(\widetilde{c}_t). \end{aligned}$$

In order to prove that $1 \implies 2$, suppose by contradiction that $Eu(\widetilde{c}_{t1}) < Eu(\widetilde{c}_{t2}) < ... < Eu(\widetilde{c}_{tn})$, but there exists $\theta \in [1, n-1]$ such that $Eu'(\widetilde{c}_{t\theta}) \leq Eu'(\widetilde{c}_{t\theta+1})$. Then, consider any increasing and concave ϕ that is locally linear for all $U \leq Eu(\widetilde{c}_{t\theta})$ and for all $U \geq Eu(\widetilde{c}_{t\theta+1})$, and has a strictly negative derivative in between these bounds. For any such function ϕ, we have that $\phi'(E\left[u(\widetilde{c}_t) \mid \widetilde{\theta}\right])$ and $E\left[u'(\widetilde{c}_{t\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic. Using the covariance rule as above, that implies that $Eu'(\overset{\circ}{\widetilde{c}_t}) < Eu'(\widetilde{c}_t)$, a contradiction.■

6.1 The CARA case

By consequence of Lemma 3, in order to sign the pessimism effect, we need to look for conditions such that u and $-u'$ indeed "agree" on a ranking of lotteries $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$. Consider first an agent who satisfies constant absolute risk aversion (CARA), i.e.

$$u(c) = -\frac{1}{A}\exp(-Ac).$$

In such a case u and $-u'$ represent identical risk preferences $-u'(c) = Au(c)$. This immediately implies the following result.

Proposition 3 *Under CARA preferences, the pessimism effect always reduces the socially efficient discount rate.*

Regardless of the specifics of the economic environment, ambiguity has an unambiguous effect on the willingness to save if risk preferences are exponential.

One might conjecture that we could extended this result beyond the CARA family. It seems natural that if u is increasing and u' is decreasing, their expectations should rank lotteries in opposite direction. Yet, the theory of stochastic dominance tells us that the solution is not that simple. Indeed, the following section reveals that the further we relax the CARA assumption, the more structure we need to impose on the marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ to recover our result.

6.2 The general case

In contrast, consider now the opposite end of the spectrum. That is, let u be an arbitrary function which satisfies our assumptions of increasingness and risk aversion. Recall that if $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to first-degree stochastic dominance (FSD), the expectation $Ef(\widetilde{c}_{t\theta})$ will be increasing in θ for all increasing functions f. Taking $f = u$ and $f = -u'$ – two increasing functions –, directly implies that condition 2 in Lemma 3 is satisfied . However, ranking the priors according to FSD is rather restrictive. It would be desirable to extend this result to a weaker stochastic order.

For instance, consider the second-degree stochastic dominance order (SSD). It guarantees that $Ef(\widetilde{c}_{t\theta})$ is increasing in θ for all increasing and concave functions f. If we assume that u has a convex derivative – that is, assuming that the representative agent is prudent–, implies that $f = -u'$ is increasing and concave. Thus, condition 2 in Lemma 3 is again satisfied in that case. This yields the following proposition.

Proposition 4 *The pessimism effect reduces the socially efficient discount rate if*

- *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to FSD.*
- *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to SSD and u exhibits prudence.*

The second sufficient condition relaxes the constraint on the structure of the ambiguity, while it constrains the set of acceptable risk attitudes. That is, it requires prudence in addition to risk aversion. In the absence of ambiguity, being prudent means that the agent would like to save more if a zero-mean risk is added to her wealth (Leland (1968), Drèze and Modigliani (1972)).

In the following proposition, we put forward a third pair of sufficient conditions. Compared to the SSD/prudence requirement, we are able to relax SSD at the cost of imposing a stronger restriction on the set of acceptable utility functions as we replace prudence by the stronger DARA condition. We will therefore employ a stochastic order introduced by Jewitt (1989).

Definition 1 *We say that $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of Jewitt if the following condition is satisfied: for all increasing and concave u, if agent u prefers $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$, then all agents more risk-averse than u also prefer $\widetilde{c}_{\theta'}$ to $\widetilde{c}_{\theta}$.*

Of course, from the definition itself, if $\widetilde{c}_{\theta'}$ dominates $\widetilde{c}_{\theta}$ in the sense of SSD, this preference order also holds in the sense of Jewitt, thereby showing that this order is weaker than SSD. Jewitt (1989) shows that distribution function $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of Jewitt if and only if the following condition holds: there exists some w in their joint support $[a, b]$, such that

$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \qquad \text{for all} \qquad x \in [a, w], \tag{14}$$

$$\int_a^w (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0 \tag{15}$$

$$\int_a^x (F_{t\theta'}(z) - F_{t\theta}(z))dz \qquad \text{is non-increasing} \quad \text{on } [w, b]. \tag{16}$$

That is, two random variables fulfill Definition 1 if there exists a consumption level w in their support such that, conditional on the outcome being lower than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of SSD, whereas conditional on the outcome being higher than w, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of FSD. Observe that second-degree stochastic dominance is indeed stronger than Jewitt's ordering, since SSD is contained in Definition 1 as a special case when we pick $w = b$.

Proposition 5 *The pessimism effect reduces the socially efficient discount rate if the set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to Jewitt's stochastic order and u exhibits decreasing absolute risk aversion.*

Proof: Decreasing absolute risk aversion means that $v = -u'$ is more concave than u in the sense of Arrow-Pratt. By definition of Jewitt's stochastic order, it implies that $Eu(\widetilde{c}_{t\theta'}) \geq Eu(\widetilde{c}_{t\theta})$ implies that $Ev(\widetilde{c}_{t\theta'}) \geq Ev(\widetilde{c}_{t\theta})$, or equivalently, that $Eu'(\widetilde{c}_{t\theta'}) \leq Eu'(\widetilde{c}_{t\theta})$. Thus Eu and Eu' are anti-comonotonic. Using Lemma 3 concludes the proof.■

Combing Lemma 1 with Propositions 3, 4 and 5 yields our main result.

Proposition 6 *Suppose that the representative agent exhibits non increasing absolute ambiguity aversion (DAAA). Then, ambiguity aversion reduces the socially efficient discount rate if one of the following conditions holds:*

1. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to FSD and u is increasing and concave.*
2. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to SSD and u is increasing, concave, and exhibits prudence.*
3. *The set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ can be ranked according to Jewitt (1989) and u is increasing and concave, and exhibits DARA.*
4. *u exhibits constant absolute risk aversion.*

Observe that the result in our analytical example in Section 3 fits condition *1*: A mere translation in the distribution constitutes a first-degree stochastic dominance. Yet, in many circumstances, the degrees of riskiness also differ across the plausible distributions, usually implying that the plausible prior distributions cannot be ranked according to FSD. Condition *2* provides a sufficient condition on risk attitudes if marginals can only be ranked according to second-degree stochastic dominance, which contains Rothschild-Stiglitz's increases in risk as a particular case. It turns out that in this case, in addition to risk-aversion, the representative agent should also be prudent. Note that even the weaker Jewitt-ordering from condition *3* only requires decreasing absolute risk aversion. This property is widely accepted in the economic literature and it is in particular compatible with the observation that more wealthy individuals tend to take more portfolio risk.[7] Finally, if one accepts DAAA and CARA, we

[7] Notice that counter-example 1, the two random variables $\widetilde{c}_{t1}$ and $\widetilde{c}_{t2}$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

provide a model-free prediction on the effect of ambiguity on the willingness to postpone consumption.

7 The comparative statics of an increase in ambiguity aversion

Our results up to now characterize the effect of smooth ambiguity aversion on the equilibrium interest rate, starting from the ambiguity-neutral benchmark. A natural question to ask is whether our results hold for any increase in ambiguity aversion.

For this purpose, consider two economies, $i = 1, 2$, which are identical up to the level of ambiguity aversion of the respective representative agent. In particular, suppose that the agent in economy 2 is more ambiguity-averse, which means that $\phi_2(U) = k(\phi_1(U))$ for all U, with $k(\cdot)$ increasing and concave. According to the adjusted pricing formula in (9) an increase in ambiguity aversion decreases the social discount rate if and only if

$$a_2 Eu'(\tilde{c}_t^2) \geq a_1 Eu'(\tilde{c}_t^1), \tag{17}$$

where a_i is defined as in (10) with ϕ being replaced by ϕ_i, and where $\tilde{c}_t^i$ is random future consumption distorted by weights q_θ^i, as in (11). Naturally, taking ϕ_1 linear, we retrieve condition (12) from the SEU benchmark.

At the outset, we are able to generalize our findings about the pessimism effect to any increase in ambiguity aversion.

Lemma 4 *The following two conditions are equivalent:*

1. *Beliefs q^2 are dominated by q^1 in the sense of the monotone likelihood ratio order for any set of marginals $(\widetilde{c}_{t1}, ..., \widetilde{c}_{tn})$ such that $Eu(\widetilde{c}_{t1}) \leq ... \leq Eu(\widetilde{c}_{tn})$.*
2. *$\phi_2 = k(\phi_1)$ is more ambiguity-averse than ϕ_1, meaning that k is increasing and concave.*

Proof: Note that we need to find that $q_{\tilde{\theta}}^2 / q_{\tilde{\theta}}^1$ and $\tilde{\theta}$ are anti-comonotonic. Using (11), we can rewrite the ratio as

$$\frac{q_\theta^2}{q_\theta^1} = k'\left(\phi_1(Eu(\widetilde{c}_{t\theta}))\right) \frac{\sum_{\tau=1}^n q_\tau \phi_1'\left(Eu(\widetilde{c}_{t\tau})\right)}{\sum_{\tau=1}^n q_\tau \phi_2'\left(Eu(\widetilde{c}_{t\tau})\right)}.$$

The fraction on the right hand side does not change with θ. Furthermore, k' is decreasing in its argument. Finally, since the argument $\phi_1(Eu(\widetilde{c}_{t\theta}))$ is itself increasing with θ by assumption, we get the desired result.■

Hence, under the stochastic order conditions from Proposition 6, more ambiguity aversion reinforces the pessimism effect, which makes saving more attractive. However, it is clear from section 5 that an increase of ambiguity aversion need not reinforce the ambiguity *prudence* effect a. In particular, introducing *increasing* absolute ambiguity aversion will in fact raise the interest rate if the representative agent is risk neutral. For small degrees of ambiguity, the impact of a change in ϕ on a depends upon its impact on the speed at which absolute ambiguity aversion decreases. This is stated in the following lemma.

Lemma 5 *Consider a family of ambiguous environments parametrized by $k \in R$ and a vector $(u_1, ..., u_n) \in R^n$ such that $Eu(\widetilde{c}_{t\theta}(k)) = u_0 + ku_\theta$ for all θ. Let us define $a(k) = \Sigma_\theta q_\theta \phi'(Eu(\widetilde{c}_{t\theta}(k)))/\phi'(V(k))$, where $\phi(V(k)) = \Sigma_\theta q_\theta \phi(Eu(\widetilde{c}_{t\theta}(k)))$. We have that*

$$a(k) = 1 - \frac{1}{2} Var(ku_{\widetilde{\theta}}) \frac{\partial}{\partial u_0} \left(\frac{-\phi''(u_0)}{\phi'(u_0)} \right) + o(k^2), \tag{18}$$

where $\lim_{k \to 0} o(k^2)/k^2 = 0$.

Proof: Observe first that $V(0) = u_0$, $V'(0) = Eu_{\widetilde{\theta}}$, and $V''(0) = Var(u_{\widetilde{\theta}})\phi''(u_0)/\phi'(u_0)$. Notice also that $a(0) = 1$. We have in turn that

$$a'(k) = \frac{E\left[u_{\widetilde{\theta}}\phi''(u_0 + ku_{\widetilde{\theta}})\right]\phi'(V(k)) - E\left[\phi'(u_0 + ku_{\widetilde{\theta}})\right]\phi''(V(k))V'(k)}{\phi'(V(k))^2}.$$

It implies that $a'(0) = 0$. Differentiating again the above equality at $k = 0$ yields

$$\begin{aligned} {\phi'_0}^2 a''(0) &= E\left[u^2_{\widetilde{\theta}}\right]\phi'_0\phi'''_0 + \left(Eu_{\widetilde{\theta}}\right)^2 \phi''^2_0 - \left(Eu_{\widetilde{\theta}}\right)^2 \phi''^2_0 - \\ &\quad - \left(Eu_{\widetilde{\theta}}\right)^2 \phi'_0\phi'''_0 - \phi'_0\phi''_0 V''(0) \\ &= \left(E\left[u^2_{\widetilde{\theta}}\right] - \left(Eu_{\widetilde{\theta}}\right)^2\right)\left(\phi'_0\phi'''_0 - \phi''^2_0\right), \end{aligned}$$

where $\phi_0^{(i)} = \phi^{(i)}(u_0)$. This implies that

$$a''(0) = -Var(u_{\widetilde{\theta}}) \frac{\partial}{\partial u_0} \left(\frac{-\phi''(u_0)}{\phi'(u_0)} \right).$$

The Taylor expansion of a yields $a(k) = a(0) + ka'(0) + 0.5k^2 a''(0) + o(k^2)$. Collecting the successive derivatives of a concludes the proof.■

A direct consequence of the above lemma is that, for small degrees of ambiguity, a_2 is larger than a_1 if and only if

$$\frac{\partial}{\partial u_0} \left(\frac{-\phi''_2(u_0)}{\phi'_2(u_0)} \right) \geq \frac{\partial}{\partial u_0} \left(\frac{-\phi''_1(u_0)}{\phi'_1(u_0)} \right), \tag{19}$$

i.e., if, locally, at the ambiguity-free expected utility level u_0, absolute ambiguity aversion decreases more rapidly under ϕ_2 than under ϕ_1. Thus, for small degrees of ambiguity, a change in the attitude towards ambiguity from ϕ_1 to ϕ_2 yields an ambiguity prudence effect that tends to reduce the interest rate if condition (19) is satisfied.

Unfortunately, as the following example shows, even if condition (19) holds for all u_0, this is not sufficient to guarantee $a_2 \geq a_1$.

Counter-example 2. Let $\phi(U) = U^{1-\eta}/(1-\eta)$ defined on R^{+}. Observe that $-\phi''(U)/\phi'(U) = \eta/U$ is positive and decreasing in its domain. Moreover, an increase in η raises both ambiguity aversion, and the speed at which absolute ambiguity aversion decreases with U. Proposition 5, yields that a is increasing in η when the risk on U is small. We show that this is not true for large degrees of ambiguity. Suppose therefore that $u(c) = c$ and that there are $n = 2$ equally likely plausible probability distributions, with $\overline{c}_1 = 0.5$ and $\overline{c}_2 = 1.5$. Suppose also that $\delta = 0.25$. In Figure 1, we draw the socially efficient discount rate r_t for $t = 1$ as a function of the degree of relative ambiguity aversion η. As stated in Proposition 2, we see that the discount rate $r_1(\eta)$ under ambiguity aversion is always smaller than under ambiguity neutrality ($r(0)$). However, the relationship between the discount rate and the degree of ambiguity aversion is not monotone. For example, increasing relative ambiguity aversion from $\eta = 3$ to any larger level *raises* the discount rate.

ftbpFU3.2932in2.034in0ptThe discount rate as a function of relative ambiguity aversion. We assume that $\phi(U) = U^{1-\eta}/(1-\eta)$, $u(c) = c$, $\delta = 0.25$, $\overline{c}_1 = 0.5$, $\overline{c}_2 = 1.5$ and $p = 0.5$.Fig1Figure

With a counter-example based on the most common family of utility functions $\phi(U) = U^{1-\eta}/(1-\eta)$, there is no hope for convincing sufficient conditions to guarantee an increase in savings. To summarize, we are left with three special cases where signing the effect on a is possible:

- The degree of ambiguity aversion is small and condition (19) is satisfied;
- The initial degree of ambiguity aversion is small, so that Proposition 2 can be used as an approximation;
- The initial ϕ_1 function exhibits non decreasing ambiguity aversion, whereas the final ϕ_2 function exhibits non increasing ambiguity aversion. This implies that $a_1 \leq 1 \leq a_2$.

Combining any of these conditions with any of the three conditions from Proposition 6 is sufficient to guarantee that a marginal increase in ambiguity aversion reduces the socially efficient discount rate.

8 Recursive smooth ambiguity preferences

The "one-step-ahead" preference model from above served to obtain accessible first-order conditions. However, it disregarded whether ambiguity would be reduced or possibly resolved in the periods before t.

Alternatively, one could follow Ju and Miao (2007) or Collard, Mukerji, Sheppard and Tallon (2008) and apply the recursive formulation of smooth ambiguity preference (KMM (2009))

$$W(c_t, h^{t-1}) = u(c_t) + \beta\phi^{-1}(\sum_{\theta=1}^{N} q_{\theta|h^t}\phi(E_{|h^t}W(\widetilde{c}_{(t+1)\theta}, h^t))$$

where $q_{\theta|h^t}$ is the probability that θ is the true parameter, given the history of observations $h^t = c_0, c_1, \ldots, c_t$, and where the agent's impatience is now represented by a factor $\beta = \frac{1}{1+\delta}$.

Take a project which matures in two periods and which pays a sure return of $(1+r)^2$ and for simplicity consider only CAAA preferences. The prior probability that the true scenario is θ equals q_θ. Seen from date 0, the utility of investing α into the project can be expressed as

$$J(\alpha) = u(c_0 - \alpha) + \beta\phi^{-1}\left(\sum_{\theta=1}^{N} q_\theta \phi(E\big[u(\widetilde{c}_{1\theta}) + \beta V(\alpha|\widetilde{c}_{1\theta})\big])\right) \tag{20}$$

where function $V(\alpha|c_1)$ represents the (distorted) continuation value of investment strategy α after observing c_1

$$\phi(V(\alpha|c_1)) = \sum_{\theta=1}^{n} q_{\theta|c_1}\phi\left(E_{|c_1}u(\widetilde{c}_{2\theta} + \alpha(1+r)^2)\right). \tag{21}$$

The critical r from condition (12) is now implicitly defined by $J'(0) = 0$. Indeed, it is easy to see that the socially efficient discount rate to be applied in period 0 is decreased by ambiguity aversion if and only if

$$\sum_{\tau=1}^{N} q_\tau^\circ E\left[\sum_{\theta=1}^{N} q^\circ_{\theta|\widetilde{c}_{1\theta}} E_{|\widetilde{c}_{1\theta}}[u'(\widetilde{c}_{2\theta})]\right] \geq \sum_{\tau=1}^{N} q_\tau E\left[\sum_{\theta=1}^{N} q_{\theta|\widetilde{c}_{1\theta}} E_{|\widetilde{c}_{1\theta}}[u'(\widetilde{c}_{2\theta})]\right]. \tag{22}$$

The first, and novel distortion q_θ° is due to the uncertainty about the continuation value at date 1, which is a function of the intermediate realization c_1. The agent thus takes into account both the consumption values of c_1 and the (discounted) effect of the "signal" c_1 on the continuation value:

$$q_\theta^\circ = q_\theta \frac{\phi'\left(E_{|c_1}\big[u(\widetilde{c}_{1\theta}) + \beta V(0|\widetilde{c}_{1\theta})\big]\right)}{\sum_{\tau=1}^{N} q_\tau \phi'\left(E_{|c_1}\big[u(\widetilde{c}_{1\tau}) + \beta V(0|\widetilde{c}_{1\tau})\big]\right)}. \tag{23}$$

However, those continuation values are again based on pessimistically distorted posteriors

$$q^\circ_{\theta|c_1} = q_{\theta|c_1}\frac{\phi'\left(E_{|c_1}u(\widetilde{c}_{2\theta})\right)}{\sum_{\tau=1}^{N} q_{\tau|c_1}\phi'\left(E_{|c_1}u(\widetilde{c}_{2\tau})\right)}.$$

The recursive model's increased complexity comes from acknowledging the multiple roles of c_1. Notice from (23) that the distortion of priors q_θ° is pessimistic with respect to a linear combination between period 1 consumption and the continuation value.

Indeed, notice from (8) that the one-step-ahead formulation and the recursive formulation agree on comparative statics if the realization c_1 is not informative about $\widetilde{c}_{2\theta}$.

The other polar case arises when the respective supports of marginals $\widetilde{c}_{1\theta}$ are disjoint. Then the realization fully reveals θ and condition (8) simplifies to

$$\sum_{\theta=1}^{N} q_\theta^\circ E_{|c_1} u'(\widetilde{c}_{2\theta}) \geq \sum_{\theta=1}^{N} q_\theta E_{|c_1} u'(\widetilde{c}_{2\theta}).$$

Accordingly, the probability measure q_θ° is clearly pessimistic if undesirable consumption scenarios in period 1 also yield a low level of expected utility in period 2, with $V(0 \mid c_1) = E_{|c_1} u(\widetilde{c}_{2\theta})$.

However, in the general case, the sufficient conditions in the recursive model are demanding. To see why, recall that $\widetilde{c}_{1\theta}$ determines not only consumption levels. Also, no matter which scenario, it's realization serves as a signal about next period's lottery through possible serial correlation. Finally, the realizations are also informative about which θ is the true one.

In order to remain consistent with the sections above, we categorize serial correlation by stochastic dominance. The following definition is taken from Gollier (2007b).

Definition 2 *The consumption process exhibits first-degree stochastic dependence (FSC) (resp. second-degree stochastic dependence (SSC)) if observing a higher value improves the posterior distribution for next period's consumption in the sense of FSD (resp. SSD).*

A simple example for FSC is an AR(1) process $\widetilde{c}_2 = \xi\widetilde{c}_1 + \widetilde{\epsilon}$, with $\xi > 0$. While a simple process with increased volatility after booms and decreased volalitility after busts, $\widetilde{c}_2 = \mu + \widetilde{c}_1\widetilde{\epsilon}$, exhibits the SSC property.[8]

The following provides sufficient conditions to rule out that good lotteries over continuation values are unfavorable in terms of intermediate consumption.

Proposition 7 *Suppose that the representative agent exhibits recursive smooth ambiguity preferences with constant absolute ambiguity aversion. Then, ambiguity aversion reduces the socially efficient discount rate if marginals can be ordered according to the monotone likelihood ratio ordering (MLR), and one of the following conditions is satisfied:*

1. *Variables $\widetilde{c}_{1\theta}$ and $\widetilde{c}_{2\theta}$ are statistically independent.*
2. *There is first-degree stochastic correlation (FSC) between $\widetilde{c}_{1\theta}$ and $\widetilde{c}_{2\theta}$.*
3. *There is second-degree stochastic correlation (SSC) between $\widetilde{c}_{1\theta}$ and $\widetilde{c}_{2\theta}$, and the agent exhibits prudence.*

[8] Gollier (2007b) determines which parameter restrictions in classical term structure models, like the one of Vasicek (1977), comply with the above taxonomy.

Proof. According to Lemma 3 $E_{|c_1}\left[u(\widetilde{c}_2) \mid \widetilde{\theta}\right]$ and $E_{|c_1}\left[u'(\widetilde{c}_{2\theta}) \mid \widetilde{\theta}\right]$ are anti-comonotonic since marginals can be ordered in the sense of MLR, which is a special case of FSD. Thus, by definition of $q^{\circ}_{\theta|c_1}$, for any realization c_1,

$$\sum_{\theta=1}^{N} q^{\circ}_{\theta|c_1} E_{|c_1} u'(\widetilde{c}_{2\theta}) \geq \sum_{\theta=1}^{N} q_{\theta|c_1} E_{|c_1} u'(\widetilde{c}_{2\theta})).$$

Hence a linear combination of c_1 on each side of the inequality must preserve the relation. In particular

$$\sum_{\tau=1}^{N} q_{\tau} E[\sum_{\theta=1}^{N} q^{\circ}_{\theta|\widetilde{c}_{1\theta}} E_{|c_1} u'(\widetilde{c}_{2\theta})] \geq \sum_{\tau=1}^{N} q_{\tau} E[\sum_{\theta=1}^{N} q_{\theta|\widetilde{c}_{1\theta}} E_{|c_1} u'(\widetilde{c}_{2\theta})].$$

Define now the distorted posterior expectations on marginal utility as a function of the signal c_1

$$M(c_1) = \sum_{\theta=1}^{N} q^{\circ}_{\theta|c_1} E_{|c_1} u'(\widetilde{c}_{2\theta}).$$

Notice that to prove inequality (8) it is sufficient to show

$$\sum_{\theta=1}^{N} q^{\circ}_{\theta} E[M(\widetilde{c}_{1\theta})] \geq \sum_{\theta=1}^{N} q_{\theta} E[M(\widetilde{c}_{1\theta})].$$

Due to the definition of distortions q°_{θ} and the MLR-ordering of marginals $\widetilde{c}_{1\theta}$, the combination of the following properties is sufficient to prove the inequality: $u(c_1)$ increasing, which is true by assumption, and $V(0|c_1)$ increasing in c_1. To complete the proof it suffices to show anti-comonotonicity, i.e. the decreasingness of $M(c_1)$.

Notice that the MLR ranking of marginals implies that posteriors $q_{\theta|c_1}$ are FSD improving along the index c_1. Thus it suffices to show that $E_{|c_1} u'(\widetilde{c}_{2\theta})$ are nonincreasing in c_1 and $E_{|c_1} u(\widetilde{c}_{2\theta})$ are nondecreasing in c_1. This is satisfied under 1. statistical indpendence, 2. increasingness of u and $-u'$ together with FSC, 3. increasingness and concavity of u and $-u'$ together with SSC. ■

Proposition 6 showed that the MLR property is not necessary to rank lotteries over consumption levels. In the recursive model however, there is a need to unambiguously identify good news. Both in terms of θ in the sense of an FSD-improved posterior for higher c_1 - guaranteed by MLR. As well as in terms of expectations about c_2 - guaranteed by restriction on serial correlation.

9 Numerical illustrations

9.1 The power-power normal-normal case

As observed in Section 3, we can solve analytically for the socially efficient discount rate by taking a "power-power" specification. That is, CRRA risk preferences and CRAA ambiguity preferences allow for an exact solution if both ambiguity and the logarithm of consumption are normally distributed. In accordance with Weitzman (2007b), who considered a similar model under ambiguity neutrality, we will establish the following parameter values as a benchmark. Consider a "quartet of twos". Namely a rate of pure preference for the present $\delta = 2\%$, a degree of relative risk aversion $\gamma = 2$, a mean growth rate of consumption $g = 2\%$, and standard deviation of growth $\sigma = 2\%$. We can rewrite the Ramsey rule (7) as

$$r_t = 5.88\% - 3\sigma_0^2 t(1 + \eta/2). \tag{24}$$

Hence, in the absence of ambiguity, the Ramsey rule prescribes a flat discount rate of 5.88%. We introduce ambiguity by assuming that the growth-trend has a normal distribution with standard deviation $\sigma_0 = 1\%$. In other words, consumers believe that with a 95% probability, the growth trend lies between 0% and 4%. Thus, even in the absence of ambiguity aversion ($\eta = 0$), the introduction of ambiguous probabilities affects the term structure of discount rates, as shown by Weitzman (2007a) and Gollier (2007b).

This is because ambiguity creates fatter tails in the distribution of future consumption. Indeed, ambiguity increases the volatility of log-consumption at date t by $\sigma_0^2 t^2$. Accordingly, the prudent agent wants to save more for the remote future, and the interest rate should fall with the time-horizon. If in addition, the agent exhibits ambiguity aversion, the social discount rate decreases more quickly, as seen in equation (24). We also infer that ambiguity has hardly any effect on the short term interest rate.

In order to calibrate the model, one needs to evaluate the degree of relative ambiguity aversion η. Consider therefore the following thought experiment.[9] Suppose that the growth rate of the economy over the next 10 years is either 20% – with probability π–, or 0%. Further, suppose that the true value of π is unknown. Rather, it is uniformly distributed on $[0, 1]$, as in the Ellsberg game in which the player has no information on the proportion of black and white balls in the urn.

Let us define the certainty equivalent growth rate $CE(\eta)$ as the sure growth rate of the economy that yields the same welfare as the ambiguous environment described above. It is implicitly defined by the following condition:

$$\left(k\frac{(1+CE)^{1-\gamma}}{1-\gamma}\right)^{1-k\eta} = \int_0^1 \left(k\left(\pi\frac{1.2^{1-\gamma}}{1-\gamma} + (1-\pi)\frac{1^{1-\gamma}}{1-\gamma}\right)\right)^{1-k\eta} d\pi,$$

where γ is set at $\gamma = 2$. In Figure **??**, we plot the certainty equivalent as a function of the degree of relative ambiguity aversion. In the absence of ambiguity aversion (or if π is known to be equal to 50%), the certainty equivalent growth rate equals $CE(0) = 9.1\%$. Surveying experimental studies, Camerer (1999) reports ambiguity premia $CE(0) - CE(\eta)$ in the order of magnitude of 10%

[9] This is based on a 10-year version of the calibration exercise performed by Collard, Mukerji, Sheppard and Tallon (2008), who considered a power-exponential specification.

of the expected value for such an Ellsberg-style uncertainty. This environment yields a reasonable ambiguity premium of 10%, i.e., a 1% reduction in the growth rate. Thus, ambiguity aversion should reduce the certainty equivalent from 9.1% to around 8%. From Figure **??**, this is compatible with a degree of relative ambiguity aversion between $\eta = 5$ and $\eta = 10$.

ftbpFU3.3529in2.0695in0ptThe certainty equivalent growth rate CE (in %) as a function of relative ambiguity aversion η. We assume that the growth rate is either 20% or 0% respectively with probability π and $1-\pi$, with $\pi \sim U(0,1)$. Relative risk aversion equals $\gamma = 2$. Fig2Figure

Table 1 reports the values of efficient rates for projects with maturity 10 and 30 respectively.

Table 1: The social discount rate at the benchmark "quartet of twos", with $\sigma_0 = 1\%$.

t	$\eta = \mathbf{0}$	$\eta = \mathbf{5}$	$\eta = \mathbf{10}$
10	5.58%	4.83%	4.08%
30	4.98%	2.73%	0.48%

While ambiguity aversion has no effect on the short term interest rate, its effect on the long rate is important. The discount rate for a cash flow occurring in 30 years is reduced from 4.98% to 2.73% when relative ambiguity aversion goes from $\eta = 0$ to $\eta = 5$.

The discrepancies between the settings call for an empirical separation between standard risk and ambiguity in an economy. While the former shifts the level of the yield curve, the latter determines its slope. A negative slope tends to increase the relative importance of long-term costs and benefits. In particular, we need to stress the amplification potential of ambiguity aversion for the evaluation of long-term projects.

9.2 An AR(1) process for log consumption with an ambiguous long-term trend

Clearly, while delivering simple expressions, our benchmark economy abstracts from rich consumption dynamics, notably any serial correlation. It is thus not surprising that our predictions do not fare well when confronted with the term structure of interest rates observed on financial markets. Thus, we will relax the assumption of uncorrelated growth rates and allow for persistence of shocks, as in Collard, Mukerji, Sheppard and Tallon (2008) and Gollier (2008). We hereafter show that this model can produce the desired non-linear term structure in the short run and the medium run. While, in the limit, it generates a linearly decreasing term structure in the long run.

Consider first an auto-regressive consumption process of order 1 à la Vasicek (1977), but in which the long-term growth μ of log consumption around which the actual growth mean-reverts is uncertain:

$$\begin{aligned}
\ln c_{t+1} &= \ln c_t + x_t \\
x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\
\varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\
\mu &\sim N(\mu_0, \sigma_0^2),
\end{aligned} \tag{25}$$

where $0 \leq \xi \leq 1$. That is, system (25) describes an AR(1) consumption process with unknown trend. The polar case without persistence ($\xi = 0$), amounts to the discrete time equivalent of the geometric Brownian motion considered in Section 3 and calibrated here above. In contrast, $\xi = 1$ describes shocks on the growth of log consumption that are fully persistent. Using the same techniques which led us to equation (6), we obtain the following generalization:

$$r_t = \delta + \gamma\frac{EX_t}{t} - \frac{1}{2}\gamma^2\frac{Var\left[X_t \mid \mu\right] + Var\left[E[X_t \mid \mu]\right]}{t} - \frac{1}{2}\eta\left|1-\gamma^2\right|\frac{Var\left[E[X_t \mid \mu]\right]}{t}, \tag{26}$$

where X_t is defined as

$$X_t = \ln c_t - \ln c_0 = \mu t + (x_{-1} - \mu)\frac{\xi(1-\xi^t)}{1-\xi} + \sum_{\tau=1}^{t}\frac{1-\xi^\tau}{1-\xi}\varepsilon_{t-\tau}.$$

It yields

$$\begin{aligned}
\frac{EX_t}{t} &= \mu_0 + (x_{-1} - \mu_0)\frac{\xi(1-\xi^t)}{t(1-\xi)}, \\
\frac{Var\left[X_t \mid \mu\right]}{t} &= \frac{\sigma^2}{(1-\xi)^2} + \sigma^2\frac{\xi(1-\xi^t)}{t(1-\xi)^3}\left[\frac{\xi(1+\xi^t)}{1+\xi} - 2\right],
\end{aligned}$$

and

$$\frac{Var\left[E[X_t \mid \mu]\right]}{t} = \frac{\sigma_0^2}{t}\left(t - \frac{\xi(1-\xi^t)}{1-\xi}\right)^2.$$

To illustrate, suppose that $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$,and $x_{-1} = 1\%$. Following Backus, Foresi and Telmer (1998), suppose also that $\xi = 0.7$ year^{-1}, such that a shock has a half-life of 3.2 years. In Figure **??**, we have drawn the term structure of discount rates for 3 different degrees of ambiguity aversion: $\eta = 0$, 5, and 10. We can see that, as in the absence of persistence, the role of ambiguity aversion is to force a downward slope of the yield curve for long time horizons. This is confirmed by the following observation:

$$\lim_{t\to\infty}\frac{\partial r_t}{\partial t} = -\frac{1}{2}\eta\left|1-\gamma^2\right|\sigma_0^2.$$

ftbpFU3.6149in2.2407in0ptThe term structure of discount rates in the case of an AR(1) process with ambiguous long-term trend, $\delta = 2\%$, $\gamma = 2$, $\mu_0 = 2\%$, $\sigma = 2\%$, $\sigma_0 = 1\%$, $x_{-1} = 1\%$, and $\xi = 0.7$.Fig3Figure

9.3 An AR(1) process for log consumption with an ambiguous degree of mean reversion

Consider alternatively an auto-regressive consumption process of order 1 with a known long-term trend, but in which there is ambiguity on the coefficient of mean reversion:

$$\begin{aligned}\ln c_{t+1} &= \ln c_t + x_t \\ x_t &= \xi x_{t-1} + (1-\xi)\mu + \varepsilon_t \\ \varepsilon_t &\sim N(0,\sigma^2),\ \varepsilon_t \perp \varepsilon_{t'} \\ \xi &\sim U(\underline{\xi},\overline{\xi}).\end{aligned}$$

There is no analytical solution for the discount rate, which must be computed numerically by estimating the following two terms, deduced from equation (3) (we normalized $c_0 = 1$):

$$\frac{E\phi'\left(Eu\right)Eu'}{u'(c_0)} = b(E\exp(G))$$

and

$$\phi'(V_t(0)) = b\left(E\exp\left(H\right)\right)^{\frac{-k\eta}{1-k\eta}},$$

where G and H correspond to

$$\begin{aligned}G &= -(\gamma + k\eta(1-\gamma))E\left[X_t \mid \xi\right] + \frac{1}{2}\left(\gamma^2 - k\eta(1-\gamma)^2\right)Var\left[X_t \mid \xi\right], \\ H &= (1-k\eta)(1-\gamma)E[X_t \mid \xi] + \frac{1}{2}(1-k\eta)(1-\gamma)^2 Var[X_t \mid \xi].\end{aligned}$$

In Figure **??** we draw the term structure of the discount rate with the same parameter values as in the previous section, except that $\mu = 2\%$ and $\xi \sim U(0.5, 0.9)$. As before, longer time horizons yields more ambiguity in the set of plausible distributions of consumption, which implies that ambiguity aversion has a stronger negative impact on the discount rates associated to these longer durations.

ftbpFU3.3684in2.0833in0ptThe term structure of discount rates in the case of an AR(1) with an ambiguous mean reversion coefficient, with $\delta = 2\%$, $\gamma = 2$, $\mu = 2\%$, $\sigma = 2\%$, $x_{-1} = 1\%$, and $\xi \sim U(0.5, 0.9)$.Fig4Figure

10 Conclusion

The present paper has shown how ambiguity-aversion changes the way one should discount future costs and benefits of investment projects. In line with recent literature, our analysis suggests that parameter uncertainty might well be decisive in long-term policy appraisals. Nevertheless, we found that, in general, it is not true that ambiguity aversion always decreases the socially efficient discount rate. We have, however, identified moderate requirements on risk-attitudes and the statistical relation among prior distributions, such that decreasing ambiguity aversion should induce us to use a smaller discount rate. Our numerical illustrations indicate that the effect of ambiguity aversion on the discount rate is large, in particular for longer time horizons.

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Appendix

Proof of Proposition 1. In order to prove this result, we need the following Lemma, which is Theorem 106 in Hardy, Littlewood and Polya (1934), Proposition 1 in Polak (1996), and Lemma 8 in Gollier (2001).

Lemma 6 *Consider a function ϕ from R to R, twice differentiable, increasing and concave. Consider a vector $(q_1, ..., q_n) \in R_+^n$ with $\sum_{j=1}^n q_j = 1$, and a function f from R^n to R, defined as*

$$f(U_1, ..., U_n) = \phi^{-1}(\sum_{\theta=1}^{n} q_\theta \phi(U_\theta)).$$

Define function T such that $T(U) = -\frac{\phi'(U)}{\phi''(U)}$. Function f is concave in R^n if and only if T is weakly concave in R.

Having established the above, consider two scalars α_1 and α_2 and let us denote $U_{i\theta} = Eu(\widetilde{c}_{t\theta} + \alpha_i e^{r_t t})$. Using the notation introduced in the Lemma, it implies that $V_t(\alpha_i) = f(U_{i1}, ..., U_{in})$. Because u is concave, we have that, for any (λ_1, λ_2) such that $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} \lambda U_{1\theta} + \lambda_2 U_{2\theta} &= E\left[\lambda_1 u(\widetilde{c}_{t\theta} + \alpha_1 e^{r_t t}) + \lambda_2 u(\widetilde{c}_{t\theta} + \alpha_2 e^{r_t t})\right] \\ &\leq Eu(\widetilde{c}_{t\theta} + \alpha_\lambda e^{r_t t}) =_{def} U_{\lambda\theta}, \end{aligned}$$

for all θ, where $\alpha_\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2$. Because f is increasing in R^n, this inequality implies that

$$\begin{aligned} V_t(\lambda_1\alpha_1 + \lambda_2\alpha_2) &= f(U_{\lambda 1}, ..., U_{\lambda n}) \\ &\geq f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}). \end{aligned} \tag{27}$$

Suppose that $-\phi'/\phi''$ be concave. By the Lemma, it implies that

$$\begin{aligned} f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1n} + \lambda_2 U_{2n}) &\geq \lambda_1 f(U_{11}, ..., U_{1n}) + \lambda_2 f(U_{21}, ..., U_{2n}) \\ &= \lambda_1 V_t(\alpha_1) + \lambda_2 V_t(\alpha_2). \end{aligned} \tag{28}$$

Combining equations (27) and (28) yields $V_t(\lambda_1\alpha_1 + \lambda_2\alpha_2) \geq \lambda_1 V_t(\alpha_1) + \lambda_2 V_t(\alpha_2)$, i.e., V_t is concave in α.■