A Extensions

In this section we consider two extensions of the baseline model. The first generalizes the model to allow for capital accumulation. This extension allows us to explore the transmission of monetary policy to asset prices, and in turn, the transmission from asset prices to the real economy. The second extension verifies the robustness of the cashless limiting results under an alternative credit arrangement where, rather than having to use the asset as collateral, investors are able to issue unsecured debt up to a given limit. As with the baseline model, we formulate these extensions in discrete time, and consider the continuous-time approximation to characterize equilibrium.

A.1 Capital accumulation

In the baseline model, the number of production units, A^s, is exogenous and constant. This means that monetary policy and the details of the OTC market structure affect asset prices but do not affect conventional measures of real economic activity, such as aggregate output or investment. In this section we endogenize the productive capacity of the economy by letting agents invest to augment the stock of productive units.

The model is as in Section 2, with the following change. We regard the productive units that yield the dividend good as a *capital stock* that can be accumulated. Specifically, in the second subperiod of period t, investors have access to a production technology that transforms $n \in \mathbb{R}_+$ units of the general good into x units of capital according to $x = f_t(n)$, where the production function f_t is strictly increasing, twice differentiable, concave, and satisfies $f_t(0) = \lim_{n\to\infty} f_t'(n) = 0$, and $f_t'(0) = \infty$. Thus, the value of an investor in the second subperiod is

$$W_t(\boldsymbol{a}_t, a_t^b, k_t) = \max_{(c_t, h_{1t}, h_{2t}, x_t, \tilde{\boldsymbol{a}}_{t+1}) \in \mathbb{R}_+^6} \left[c_t - h_t + \beta \mathbb{E}_t \int V_{t+1}\left(\boldsymbol{a}_{t+1}, \varepsilon\right) dG(\varepsilon) \right]$$

$$\text{s.t. } c_t + \boldsymbol{\phi}_t \tilde{\boldsymbol{a}}_{t+1} \leq h_{1t} + \boldsymbol{\phi}_t \boldsymbol{a}_t + a_t^b - k_t + \phi_t^s x_t + T_t,$$

with $\boldsymbol{a}_{t+1} = (\tilde{a}_{t+1}^m, \eta \tilde{a}_{t+1}^s)$, $h_t = h_{1t} + h_{2t}$, and $x_t = f_t(h_{2t})$, where h_t is the labor input (effort) devoted to production of general goods (equal to the quantity of general goods produced), h_{1t} is the quantity of general goods used for consumption, and h_{2t} is the quantity of general goods used as input to produce new capital, x_t. This problem can be written as

$$
\begin{aligned}
W_t(\boldsymbol{a}_t, a_t^b, k_t) = {} & \boldsymbol{\phi}_t \boldsymbol{a}_t + a_t^b - k_t + T_t \\
& + \max_{\tilde{\boldsymbol{a}}_{t+1} \in \mathbb{R}_+^2} \left[-\boldsymbol{\phi}_t \tilde{\boldsymbol{a}}_{t+1} + \beta \mathbb{E}_t \int V_{t+1}\left(\boldsymbol{a}_{t+1}, \varepsilon\right) dG(\varepsilon) \right] \\
& + \max_{h_{2t} \in \mathbb{R}_+} \left[\phi_t^s f_t\left(h_{2t}\right) - h_{2t} \right].
\end{aligned}
$$

This value function is the same as (3), except for the addition of the last term that represents the investor's profit from producing and selling new capital at the market price ϕ_t^s. The optimal quantity of general goods that the investor devotes to the production of capital goods, i.e., the h_{2t} that satisfies $\phi_t^s f_t'\left(h_{2t}\right) = 1$, is denoted $g_t\left(\phi_t^s\right)$, i.e.,

$$
g_t\left(\phi_t^s\right) \equiv f_t'^{-1}\left(1/\phi_t^s\right). \tag{62}
$$

The quantity of new capital created by an individual investor is $x_t\left(\phi_t^s\right) \equiv f_t\left(g_t\left(\phi_t^s\right)\right)$. We can regard $x_t\left(\phi_t^s\right)$ as an individual investor's contribution to aggregate investment; aggregate capital investment is

$$
X_t\left(\phi_t^s\right) \equiv x_t\left(\phi_t^s\right) N_I. \tag{63}
$$

The assumptions on f imply aggregate investment is increasing in the market price of the equity shares of capital, i.e.,

$$
X_t'\left(\phi_t^s\right) = -\frac{f_t'\left(g_t\left(\phi_t^s\right)\right)}{\left(\phi_t^s\right)^2 f_t''\left(g_t\left(\phi_t^s\right)\right)} N_I > 0.
$$

The law of motion of the aggregate capital stock is

$$
A_{t+1}^s = \eta\left(A_t^s + X_t\right), \tag{64}
$$

where X_t is aggregate investment added to the capital stock at the end of period t, and $A_0^s \in \mathbb{R}_+$ is given.[38]

The definition of equilibrium for the economy with capital accumulation is the same as Definition 1, but with two additional equilibrium variables, namely $\left\{X_t, A_{t+1}^s\right\}_{t=0}^{\infty}$, and two additional equilibrium conditions, namely $X_t = X_t\left(\phi_t^s\right)$ and (64). A RNE is a nonmonetary equilibrium with the structure described in Definition 2. A RME is a monetary equilibrium in which: (i) real equity prices (general goods per equity share) are time-invariant linear functions

[38]Since agents can now augment the stock of productive units, the beginning-of-period exogenous lump-sum endowment is no longer needed to offset the depreciation in the aggregate capital stock due to the idiosyncratic obsolesence shock that affects each individual unit of capital.

of the aggregate dividend, i.e., $\phi_t^s = \phi^s y_t$, $p_t \phi_t^m \equiv \bar{\phi}_{mt}^s = \bar{\phi}_m^s y_t$, and $p_t/q_t \equiv \bar{\phi}_{bt}^s = \bar{\phi}_b^s y_t$ for some $\phi^s, \bar{\phi}_m^s, \bar{\phi}_b^s \in \mathbb{R}_+$; and (*ii*) real money balances are a constant proportion of aggregate output, i.e., $\phi_t^m A_t^m = Z A_t^s y_t$ for some $Z \in \mathbb{R}_{++}$. Hence in a RME, $\varepsilon_t^* = (p_t \phi_t^m - \phi_t^s) \frac{1}{y_t} = \bar{\phi}_m^s - \phi^s \equiv \varepsilon^*$, $\varepsilon_t^{**} = (p_t/q_t - \phi_t^s) \frac{1}{y_t} = \bar{\phi}_b^s - \phi^s \equiv \varepsilon^{**}$, and nominal prices are

$$p_t = (\varepsilon^* + \phi^s) \frac{A_t^m}{Z A_t^s} \tag{65}$$

$$\phi_t^m = \frac{Z A_t^s y_t}{A_t^m} \tag{66}$$

$$q_t = \frac{\varepsilon^* + \phi^s}{\varepsilon^{**} + \phi^s} \frac{A_t^m}{Z A_t^s y_t}. \tag{67}$$

For the analysis that follows, we generalize the money supply process of Section 2 to the following money-growth rule

$$\frac{A_{t+1}^m}{A_t^m} = \frac{A_{t+1}^s}{A_t^s} \mu. \tag{68}$$

Notice that just as in the model of Section 2, this monetary policy rule implies the gross inflation rate (as measured by the growth in the nominal price of equity shares) is constant and equal to μ, i.e., $p_{t+1}/p_t = \mu$. In the special case with $A_t^s = A^s$, (68) reduces to $A_{t+1}^m / A_t^m = \mu$, namely the money growth process in our baseline model.

In a recursive equilibrium (monetary or nonmonetary), once the asset price ϕ^s has been found, aggregate investment is given by $X_t = X_t(\phi^s y_t)$, and the aggregate capital stock follows the stochastic difference equation $A_{t+1}^s = \eta [A_t^s + X_t(\phi^s y_t)]$. If the equilibrium is monetary, once $(\varepsilon^*, \varepsilon^{**}, \phi^s, Z)$ have been found, the implied equilibrium stochastic processes for the nominal prices, $\{p_t, \phi_t^m, q_t\}$, are given by (65), (66), and (67). Thus, along a RME, $(\varepsilon^*, \varepsilon^{**}, \phi^s, Z, \chi_{11})$ are constant, while nominal prices $\{p_t, \phi_t^m, q_t\}$ are random variables whose evolutions over time are driven by the stochastic dividend process $\{y_t\}$, and possibly also by transitional dynamics.[39] To streamline the presentation, we assume

$$f_t(n) = \varpi_t n^\sigma, \text{ with } \sigma \in (0,1), \text{ and } \varpi_t \equiv (\sigma y_t)^{-\sigma}. \tag{69}$$

Let $X_t(\Delta)$ denote aggregate investment in the recursive equilibrium of the discrete-time economy where the length of the time period is Δ, and let $\mathcal{X}_t \equiv \lim_{\Delta \to 0} \frac{1}{\Delta} X_t(\Delta)$ (i.e., $\mathcal{X}_t$ is the

[39]By way of example, notice that if $f_t = f$ and $y_t = y$ for all t, then (62) implies aggregate investment is constant, i.e., $X_t(\phi^s y_t) = X(\phi^s y)$ for all t, and A_{t+1}^s converges monotonically to the unique steady state $\bar{A}^s = \frac{\eta}{1-\eta} X(\phi^s y)$ from any initial condition A_0^s. Given the deterministic transition path $A_{t+1}^s = \eta [A_t^s + X(\phi^s y)]$, the money supply process $\{A_t^m\}$, and nominal prices $\{p_t, \phi_t^m, q_t\}$, just follow (68), (65), (66), and (67).

investment rate). Then as $\Delta \to 0$, (64) can be approximated by

$$\dot{A}_t^s = \mathcal{X}_t - \delta A_t^s. \tag{70}$$

Next, we characterize the RNE and RME for the limiting economy as $\Delta \to 0$.

Proposition 7 *Consider the limiting economy (as $\Delta \to 0$) with capital accumulation, and $\alpha \in (0,1)$.*

(i) There exists a unique recursive nonmonetary equilibrium, $(\varepsilon^n, \varphi^n, \mathcal{X}^n)$. Moreover, $(\varepsilon^n, \varphi^n)$ are as described in Proposition 1, aggregate investment rate is $\mathcal{X}^n = (\varphi^n/\rho)^{\frac{\sigma}{1-\sigma}} N_I$, and the capital stock follows (70) with $\mathcal{X}_t = \mathcal{X}^n$.

(ii) If $0 \leq \iota < \bar{\iota}(\lambda)$, there exists a unique recursive monetary equilibrium, $(\varepsilon^*, \varepsilon^{**}, \chi, \varphi, \mathcal{Z}, \mathcal{X})$. Moreover, $(\varepsilon^*, \varepsilon^{**}, \chi, \varphi, \mathcal{Z})$ are as described in Proposition 2, aggregate investment rate is $\mathcal{X} = (\varphi/\rho)^{\frac{\sigma}{1-\sigma}} N_I$, and the capital stock follows (70) with $\mathcal{X}_t = \mathcal{X}$.

Proposition 7 delivers a link between the asset price, i.e., the relative price of capital in terms of consumption goods (φ^n in the RNE or φ in the RME), and aggregate investment.

A.2 Unsecured credit

In our baseline formulation, we modeled credit in the form of margin loans mainly because it is the most common form of credit used in financial markets. In this section we verify the robustness of our main results to an alternative credit arrangement where in the OTC round, investors are able to issue unsecured debt up to a given limit. The only relevant difference in the model is that the last constraint in the bargaining problem (2) is replaced by

$$-\bar{B}_t \leq \overline{a}_t^b, \tag{71}$$

where $\bar{B}_t \geq 0$ is the credit limit faced by an individual agent in the OTC round of period t. Suppose a broker extends an investor a loan of L dollars in order to purchase A dollars worth of an asset. Suppose, as will be the case in the model, that the investor chooses to borrow the maximum amount possible, i.e., $L = \bar{B}_t$. In this case, the margin is $\mathcal{M} = 1 - \bar{B}_t/A$, leverage is $\mathcal{L} = A/\left(A - \bar{B}_t\right)$, and the loan-to-value ratio is $\mathcal{R} = \bar{B}_t/A$.

We focus on recursive equilibria. To this end, let

$$\bar{B}_t \equiv \Lambda \frac{(p_t/q_t)\, A^s}{N_I} \tag{72}$$

for some $\Lambda > 0$. In a nonmonetary economy, $\bar{\phi}_t^s \equiv p_t/q_t$, and therefore (72) amounts to assuming $\bar{B}_t \equiv \Lambda \frac{\bar{\phi}_t^s A^s}{N_I}$. Formulation (72) corresponds to an economy (monetary or nonmonetary) where the aggregate real borrowing capacity of investors expressed in terms of general goods, i.e., $N_I \bar{B}_t$, is a multiple Λ of the real value (expressed in terms of general goods) of the equity shares outstanding, $(p_t/q_t)\, A^s$.

The structure of the recursive equilibrium is as described in Definition 2 and Definition 3. We again consider the limiting economy as $\Delta \to 0$, and as before, let $\varphi \equiv \rho \phi^s$, $\mathcal{Z} \equiv \rho Z$, and $\iota \equiv i^p/\rho$. For the following result it is convenient to define

$$\bar{\varsigma}_0 \equiv \frac{\bar{\varepsilon} - \varepsilon_L + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)}{\bar{\varepsilon} + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)}$$

$$\hat{\varsigma}_0 \equiv \frac{\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)}{\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)},$$

where $\varepsilon^n \in [\varepsilon_L, \varepsilon_H]$ is the unique solution to

$$G(\varepsilon^n) = \frac{\Lambda}{1+\Lambda}.$$

Proposition 8 *Consider the limiting economy (as $\Delta \to 0$) with individual borrowing limit (72), and $\alpha \in [0,1]$. Let*

$$\varphi_0^n \equiv \lim_{\alpha \to 0} \varphi^n = \bar{\varepsilon} + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon).$$

As $\alpha \to 0$,

(i) If $\hat{\varsigma}_0 < \iota < \bar{\varsigma}_0$, then

$$\frac{\mathcal{Z}}{\varphi} \to 0$$
$$\mathcal{V} \to \infty$$
$$\varphi \to \varphi_0^n + (1-\theta) \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon),$$

where $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ is the unique solution to

$$\frac{(1-\theta) \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) + \theta \left[(\varepsilon^n - \varepsilon^*)\, G(\varepsilon^n) + \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)\right]}{\bar{\varepsilon} + (1-\theta) \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)} = \iota.$$

(ii) If $0 < \iota \leq \hat{\varsigma}_0$, then

$$\frac{\mathcal{Z}}{\varphi} \to \frac{G(\varepsilon^*) - [1 - G(\varepsilon^*)]\,\Lambda}{1 - G(\varepsilon^*)}$$
$$\mathcal{V} \to \frac{G(\varepsilon^*)\,[1 - G(\varepsilon^*)]}{G(\varepsilon^*) - [1 - G(\varepsilon^*)]\,\Lambda}$$
$$\varphi \to \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon)\,,$$

where $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ is the unique solution to

$$\frac{\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)}{\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon)} = \iota.$$

Proposition 8, which is analogous to Proposition 6, considers the limiting economy as the fraction of investors who do not have access to margin loans vanishes.

For the limiting economy as $\alpha \to 0$, Figure 3 illustrates in the space of parameters ι (vertical axis) and Λ (horizontal axis), the regions where the equilibria described in parts (i) and (ii) of Proposition 8 exist. Notice that $\bar{\varsigma}_0$ and $\hat{\varsigma}_0$ are functions of ε^n, which is in turn a function of Λ, so to make this dependence explicit, we can write $\bar{\varsigma}_0(\Lambda)$ and $\hat{\varsigma}_0(\Lambda)$. The boundaries $\iota = \bar{\varsigma}_0(\Lambda)$ and $\iota = \hat{\varsigma}_0(\Lambda)$ define three regions.[40] First, if the nominal policy rate is very high, i.e., if $\bar{\varsigma}_0(\Lambda) \leq \iota$, then the monetary equilibrium does not exist. Second, if the nominal policy rate is relatively low, i.e., if $0 < \iota \leq \hat{\varsigma}_0(\Lambda)$ as in part (ii) of Proposition 8, then the aggregate money demand from investors without access to credit vanishes in the limit, but the aggregate money demand from investors with access to credit who have low valuation remains positive in the limit, and therefore real balances and velocity converge to positive limits. Third, if the nominal policy rate is relatively high, i.e., if $\hat{\varsigma}_0(\Lambda) < \iota < \bar{\varsigma}_0(\Lambda)$ as in part (i) of Proposition 8, then real balances converge to zero and transaction velocity diverges to infinity as $\alpha \to 0$. The economic rationale for these results is as explained in the context of Proposition 6. And again, the key

[40]It is easy to prove that $\hat{\varsigma}_0(\Lambda) \leq \bar{\varsigma}_0(\Lambda)$ for all $\Lambda \geq 0$ (with "=" only if $\Lambda = 0$), and that

$$\lim_{\Lambda \to \infty} \hat{\varsigma}_0(\Lambda) = 0 < \bar{\varsigma}_0(0) = \hat{\varsigma}_0(0) = \frac{\bar{\varepsilon} - \varepsilon_L}{\bar{\varepsilon}} \leq \lim_{\Lambda \to \infty} \bar{\varsigma}_0(\Lambda) = \frac{\theta(\varepsilon_H - \varepsilon_L) + (1-\theta)(\bar{\varepsilon} - \varepsilon_L)}{\bar{\varepsilon} + \theta(\varepsilon_H - \bar{\varepsilon})},$$

where the second inequality is strict unless $\theta = 0$. Also,

$$\frac{\partial \hat{\varsigma}_0}{\partial \varepsilon^n} < 0 \leq \frac{\theta G(\varepsilon^n)\,\varepsilon_L}{\left[\bar{\varepsilon} + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)\right]^2} = \frac{\partial \bar{\varsigma}_0}{\partial \varepsilon^n}.$$

Hence, $\frac{d\hat{\varsigma}_0}{d\Lambda} < 0 \leq \frac{d\bar{\varsigma}_0}{d\Lambda}$.

result is that even though real balances and velocity converge to their nonmonetary equilibrium levels as $\alpha \to 0$, the real equity price in this cashless limit of the monetary economy exceeds the (corresponding limit of the) nonmonetary-equilibrium price by the value of a resale-option term. Since ε^* is a function of ι, the asset price is still responsive to monetary policy in the cashless limit, and that the magnitude of this response remains bounded away from zero even though real balances converge to zero.

B Efficiency and welfare

In this section we pose and solve the planner problems corresponding to the baseline model with fixed capital of Section 2, and the model with capital accumulation of Section A.1. In both cases we consider a social planner who wishes to maximize the sum of all agents' expected discounted utilities subject to the same meeting frictions that individual agents face in the decentralized formulation. Specifically, in the first subperiod of every period, the planner can reallocate assets among all investors. We restrict attention to symmetric allocations (identical agents receive equal treatment). For each of the two economies, we also provide a measure of welfare along an equilibrium path, based on the (equally weighted) sum of all agents' expected discounted utilities at the beginning of a period.

B.1 Endowment economy

Let c_t^k and h_t^k denote consumption and production of the homogeneous consumption good in the second subperiod of period t of an agent of type $k \in \{B, I\}$. Let $\tilde{a}_t^I$ denote the beginning-of-period t (before depreciation) equity holding of an individual investor. Let $\bar{a}_t^I$ denote a measure on $\mathcal{F}([\varepsilon_L, \varepsilon_H])$, the Borel σ-field defined on $[\varepsilon_L, \varepsilon_H]$. The measure $\bar{a}_t^I$ is interpreted as the distribution of post-OTC-trade asset holdings among investors with different valuations in the first subperiod of period t.

With this notation, and letting

$$\Pi \equiv \left\{\tilde{a}_{t+1}^I, \bar{a}_t^I, [c_t^k, h_t^k]_{k\in\{B,I\}}\right\}_{t=0}^{\infty},$$

the planner's problem for the model with fixed capital is

$$W^*(y_0) = \max_{\Pi} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}_t^I(d\varepsilon) N_I + \sum_{k\in\{B,I\}} \left(c_t^k - h_t^k\right) N_k \right\} \tag{73}$$

subject to

$$\tilde{a}_t^I N_I \leq A^s \tag{74}$$

$$\int_{\varepsilon_L}^{\varepsilon_H} \bar{a}_t^I (d\varepsilon) \leq a_t^I \tag{75}$$

$$\sum_{k\in\{B,I\}} \left(c_t^k - h_t^k\right) N_k \leq 0 \tag{76}$$

$$a_t^I N_I = \eta \tilde{a}_t^I N_I + (1-\eta) A^s, \tag{77}$$

and subject to the allocation Π being nonnegative (the expectation operator $\mathbb{E}_0$ is with respect to the probability measure induced by the dividend process). The following proposition characterizes the efficient allocation and the maximum value of the planner's problem.

Proposition 9 *Consider the limiting economy (as $\Delta \to 0$) with exogenous capital. The efficient allocation is characterized by $\bar{a}_t^I(E) = \frac{A^s}{N_I}\mathbb{I}_{\{\varepsilon_H \in E\}}$ for all t, where $\mathbb{I}_{\{\varepsilon_H \in E\}}$ is an indicator function that takes the value 1 if $\varepsilon_H \in E$, and 0 otherwise, for any $E \in \mathcal{F}([\varepsilon_L, \varepsilon_H])$. The welfare achieved by the planner is*

$$\mathcal{W}^*(y_t) = \frac{\varepsilon_H}{r-g} A^s y_t. \tag{78}$$

According to Proposition 9, the optimal allocation is characterized by the following simple property: only those investors with the highest valuation hold equity shares at the end of the OTC round of trade. In this context, ε_H can be interpreted as the (flow) shadow value of the asset for the planner, i.e., it is the analogue of φ^n in Proposition 1 or φ in Proposition 2. Recall that, $\varphi^n \leq \varphi \leq \varepsilon_H$ (part (i) of Proposition 3).

In Appendix D (part (i) of Lemma 27), we show that along the path of a RNE of the limiting continuous-time economy with exogenous capital, welfare is

$$\mathcal{V}^n(y_t) = \frac{\varphi_1^n}{r-g} A^s y_t, \tag{79}$$

where

$$\varphi_1^n \equiv \bar{\varepsilon} + (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right], \tag{80}$$

and ε^n satisfies (33). Notice that φ_1^n is the stock price in the RNE of an economy with $\theta = 1$.

In D (part (ii) of Lemma 27), we show that along the path of a RME of the limiting continuous-time economy with exogenous capital, welfare is

$$\mathcal{V}^m(\mathcal{Z}, y_t) = \frac{1}{r-g}\left(u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1\right) A^s y_t, \tag{81}$$

where

$$u_1^z \equiv \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) + (1-\alpha)\left[\varepsilon^{**} - \varepsilon^* + \frac{1}{1-\lambda}\int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right] \tag{82}$$

$$u_1^s \equiv \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon)$$
$$+ (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right], \tag{83}$$

$(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$ satisfy the equilibrium conditions in Proposition 2, and

$$\varphi_1 \equiv \bar{\varepsilon} + u_1^s = \varepsilon^* + u_1^z \tag{84}$$

is the normalized (i.e., multiplied by ρ) stock price in the RME of an economy with $\theta = 1$.

The following result is a corollary of Proposition 2, part (i) of Proposition 3, Proposition 9, Proposition 11, (79), and (81).

Corollary 1 *Consider the limiting economy (as $\Delta \to 0$) with exogenous capital stock. Then*

$$\mathcal{V}^n(y_t) \le \mathcal{V}^m(\mathcal{Z}, y_t) \le \mathcal{W}^*(y_t),$$

where the first inequality is strict unless $\iota = \bar{\iota}(\lambda)$, and the second inequality is strict unless $\iota = 0$.

The following result, a corollary of (81)-(83) and Lemma 6, describes welfare in the limiting economy with exogenous capital where all agents have access to credit.

Corollary 2 *Consider the limiting economy (as $\Delta \to 0$) with exogenous capital stock, with $\alpha \in [0,1]$ and $\lambda \in (0,1]$. As $\alpha \to 0$,*

(i) If $\hat{\varsigma}(0) < \iota < \bar{\varsigma}(0)$, then

$$\lim_{\alpha \to 0} \mathcal{V}^m(\mathcal{Z}, y_t) = \lim_{\alpha \to 0} \mathcal{V}^n(y_t) = \frac{\tilde{\varphi}_1^n}{r-g} A^s y_t.$$

where

$$\tilde{\varphi}_1^n \equiv \lim_{\alpha \to 0} \varphi_1^n = \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon). \tag{85}$$

(ii) If $0 < \iota \le \hat{\varsigma}(0)$, then

$$\lim_{\alpha \to 0} \mathcal{V}^n(y_t) < \lim_{\alpha \to 0} \mathcal{V}^m(\mathcal{Z}, y_t) = \frac{\tilde{\varphi}_1^z}{r-g} A^s y_t,$$

where

$$\tilde{\varphi}_1^z \equiv \tilde{\varphi}_1 + \tilde{u}_1^z \frac{G(\varepsilon^*) - \lambda}{1 - G(\varepsilon^*)} \tag{86}$$

$$\tilde{\varphi}_1 \equiv \bar{\varepsilon} + \tilde{u}_1^s \tag{87}$$

with

$$\tilde{u}_1^z \equiv \frac{1}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \tag{88}$$

$$\tilde{u}_1^s \equiv \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon), \tag{89}$$

and ε^* *satisfies (54).*

Part (i) of Corollary 2 corresponds to the parametrizations characterized in part (i) Proposition 6 for which the limiting economy as $\alpha \to 0$, is cashless. In this case, welfare in the cashless limit of the monetary economy equals welfare in the nonmonetary equilibrium. Thus, although monetary policy affects the stock price in the cashless limit (part (i) of Proposition 6), it does not affect welfare, which is identical to welfare in an economy with no money. This result is in part due to the fact that, since the capital stock, A^s, is exogenous, changes in the market price of capital, ϕ_t^s, have no effect on the allocation of resources in the cashless limit. This result, however, is driven by the fact that the capital stock is exogenous in this formulation.

B.2 Economy with capital accumulation

Next, we turn to the efficient allocation for the economy with capital accumulation. As in Proposition 7, we continue to assume (69). The notation for the planner's problem is as before, except that now we use h_{1t}^I to denote the quantity of general goods used for consumption, h_{2t}^I to denote the quantity of general goods used as input to produce new capital, and $h_t^I = h_{1t}^I + h_{2t}^I$ to denote the labor input (effort) devoted to production of general goods (equal to the quantity of general goods produced). In this case, letting

$$\Pi \equiv \left\{\tilde{a}_{t+1}^I, c_t^I, h_{1t}^I, h_{2t}^I, \bar{a}_t^I, X_t, c_t^B, h_t^B\right\}_{t=0}^{\infty},$$

the planner's problem is

$$W^*(A_0, y_0) = \max_{\Pi} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}_t^I (d\varepsilon) N_I + \sum_{k \in \{B,I\}} \left(c_t^k - h_t^k\right) N_k \right\} \tag{90}$$

subject to

$$\tilde{a}^I_{t+1} N_I \leq A^s_t + X_t \tag{91}$$

$$a^I_{t+1} N_I = A^s_{t+1} = \eta \left(A^s_t + X_t \right) \tag{92}$$

$$X_t = f_t \left(h^I_{2t} \right) N_I \tag{93}$$

$$\sum_{k \in \{B,I\}} c^k_t N_k \leq h^I_{1t} N_I + h^B_t N_B \tag{94}$$

$$\int_{\varepsilon_L}^{\varepsilon_H} \bar{a}^I_t \left(d\varepsilon \right) \leq a^I_t , \tag{95}$$

and subject to the allocation Π being nonnegative. Let $\mathcal{X}^*$ denote optimal aggregate investment rate (i.e., $\mathcal{X}^* \Delta$ is optimal investment over a time interval of length Δ).

Proposition 10 *Consider the limiting economy (as $\Delta \to 0$) with capital accumulation. The efficient allocation is characterized by the following conditions: (i) $\bar{a}^I_t \left(E \right) = \frac{A^s_t}{N_I} \mathbb{I}_{\{\varepsilon_H \in E\}}$ for all t, where $\mathbb{I}_{\{\varepsilon_H \in E\}}$ is an indicator function that takes the value 1 if $\varepsilon_H \in E$, and 0 otherwise, for any $E \in \mathcal{F} \left([\varepsilon_L, \varepsilon_H] \right)$; (ii) $\mathcal{X}^* = \left(\varepsilon_H / \rho \right)^{\frac{\sigma}{1-\sigma}} N_I$ for all t; and (iii) the capital stock follows (70) with $\mathcal{X}_t = \mathcal{X}^*$. The welfare achieved by the planner is*

$$\mathcal{W}^* \left(A^s_t , y_t \right) = \left[\frac{\varepsilon_H}{\rho} A^s_t + \frac{1}{r-g} \left(1 - \sigma \right) \left(\frac{\varepsilon_H}{\rho} \right)^{\frac{1}{1-\sigma}} N_I \right] y_t . \tag{96}$$

In the setup with capital accumulation the planner optimizes along two margins: the reallocation of the asset, and the investment margin. Optimal reallocation in the OTC trading round is as in the model with exogenous capital, while the optimal investment decision involves equating the marginal rate of substitution between labor and general goods to the optimal shadow price of capital, ε_H.

Next, we characterize the welfare function for the economy of Section A.1). As in Proposition 7, we assume (69).

In Appendix D (part (i) of Lemma 28), we show that along the path of a RNE of an economy with capital accumulation, welfare is

$$\mathcal{V}^n \left(A^s_t , y_t \right) = \left[\frac{\varphi^n_1}{\rho} A^s_t + \frac{1}{r-g} \left(\frac{\varphi^n_1}{\varphi^n} - \sigma \right) \left(\frac{\varphi^n}{\rho} \right)^{\frac{1}{1-\sigma}} N_I \right] y_t , \tag{97}$$

where φ^n is given by (32) (with ε^n given by (33)), φ^n_1 is given in (80), and the capital stock follows (70) with $\mathcal{X}_t = \left(\varphi^n / \rho \right)^{\frac{\sigma}{1-\sigma}} N_I$.

In Appendix D (part (*ii*) of Lemma 28), we show that along the path of a RME of an economy with capital accumulation, welfare is

$$\mathcal{V}^m\left(\mathcal{Z}, A_t^s, y_t\right) = \left[\frac{\varphi_1^z}{\rho} A_t^s + \frac{1}{r-g}\left(\frac{\varphi_1^z}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right] y_t, \tag{98}$$

where

$$\varphi_1^z \equiv \varphi_1 + u_1^z \frac{\mathcal{Z}}{\varphi},$$

u_1^z and u_1^s are given by (82) and (83), φ_1 is given in (84), the capital stock follows (70) with $\mathcal{X}_t = (\varphi/\rho)^{\frac{\sigma}{1-\sigma}} N_I$, and $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$ satisfy the equilibrium conditions in Proposition 2.

The following result is a corollary of Proposition 2, part (*i*) of Proposition 3, Proposition 10, Proposition 11, (97), and (98).

Corollary 3 *Consider the limiting economy (as $\Delta \to 0$) with capital accumulation and initial condition $(A_t^s, y_t) = (A_0^s, y_0)$. Then*

$$\mathcal{V}^n\left(A_0^s, y_0\right) \leq \mathcal{V}^m\left(\mathcal{Z}, A_0^s, y_0\right) \leq \mathcal{W}^*\left(A_0^s, y_0\right),$$

where the first inequality is strict unless $\iota = \bar{\iota}(\lambda)$, and the second inequality is strict unless $\iota = 0$.

The following result, a corollary of (98) and Lemma 6, describes welfare in the limiting economy with capital accumulation where all agents have access to credit.

Corollary 4 *Consider the limiting economy with capital accumulation (as $\Delta \to 0$) and initial condition $(A_t^s, y_t) = (A_0^s, y_0)$, with $\alpha \in [0,1]$ and $\lambda \in (0,1]$. As $\alpha \to 0$,*

$$\lim_{\alpha \to 0} \mathcal{V}^n\left(A_0^s, y_0\right) = \left[\frac{\tilde{\varphi}_1^n}{\rho} A_0^s + \frac{1}{r-g}\left(\frac{\tilde{\varphi}_1^n}{\tilde{\varphi}^n} - \sigma\right)\left(\frac{\tilde{\varphi}^n}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right] y_0,$$

with $\tilde{\varphi}^n$ and $\tilde{\varphi}_1^n$ given by (46) and (85). Moreover,

(i) If $\hat{\varsigma}(0) < \iota < \bar{\varsigma}(0)$, then

$$\begin{aligned} 0 &\leq \lim_{\alpha \to 0}\left[\mathcal{V}^m\left(\mathcal{Z}, A_0^s, y_0\right) - \mathcal{V}^n\left(A_0^s, y_0\right)\right] \\ &= \frac{1}{r-g}\left[\left(\frac{\tilde{\varphi}_1^n}{\tilde{\varphi}} - \sigma\right)\left(\frac{\tilde{\varphi}}{\rho}\right)^{\frac{1}{1-\sigma}} - \left(\frac{\tilde{\varphi}_1^n}{\tilde{\varphi}^n} - \sigma\right)\left(\frac{\tilde{\varphi}^n}{\rho}\right)^{\frac{1}{1-\sigma}}\right] N_I y_0, \end{aligned}$$

with "<" if $\theta \in [0,1)$, where $\tilde{\varphi}$ given by (49) (with ε^n given by (33), and ε^* given by (50)).

(ii) If $0 < \iota \leq \hat{\varsigma}(0)$,

$$
\begin{aligned}
0 &< \lim_{\alpha \to 0} \left[\mathcal{V}^m\left(\mathcal{Z}, A_0^s, y_0\right) - \mathcal{V}^n\left(A_0^s, y_0\right)\right] \\
&= \left\{\frac{1}{\rho}\left(\tilde{\varphi}_1^z - \tilde{\varphi}_1^n\right) A_0^s + \frac{1}{r-g}\left[\left(\frac{\tilde{\varphi}_1^z}{\tilde{\varphi}} - \sigma\right)\left(\frac{\tilde{\varphi}}{\rho}\right)^{\frac{1}{1-\sigma}} - \left(\frac{\tilde{\varphi}_1^n}{\tilde{\varphi}^n} - \sigma\right)\left(\frac{\tilde{\varphi}^n}{\rho}\right)^{\frac{1}{1-\sigma}}\right] N_I\right\} y_0,
\end{aligned}
$$

where

$$\tilde{\varphi}_1^z \equiv \tilde{u}_1^z \frac{G\left(\varepsilon^*\right) - \lambda}{1 - G\left(\varepsilon^*\right)} + \tilde{\varphi}_1,$$

with $\tilde{\varphi}$ given by (53) (with ε^* given by (54)), $\tilde{\varphi}_1$ given by (87), and $\tilde{u}_1^z$ given by (88).

In Corollary 4, the thought experiment consists of taking the limit as $\alpha \to 0$ while keeping the initial capital stock, A_0^s, the same. Part (i) is an economy where money is dominated in rate of return by bonds (i.e., it corresponds to part (i) of Proposition 6). In this case, for a given A_0^s, welfare in the cashless limit of the monetary economy is strictly higher than in the nonmonetary economy, provided $\theta < 1$. In the cashless limit of the monetary economy the equilibrium capital stock is $A_t^s = e^{-\delta t} A_0^s + \left(1 - e^{-\delta t}\right)\left(\tilde{\varphi}/\rho\right)^{\frac{\sigma}{1-\sigma}} N_I/\delta$, while in the nonmonetary economy the capital stock is $A_t^s = e^{-\delta t} A_0^s + \left(1 - e^{-\delta t}\right)\left(\tilde{\varphi}^n/\rho\right)^{\frac{\sigma}{1-\sigma}} N_I/\delta$, so aggregate consumption (of the dividend good), $C_t = y_t A_t^s$, is higher in the former.

C Monetary policy, asset prices, and real activity

In this section we study the effects of monetary policy on asset prices and real activity. We first characterize optimal monetary policy, and then turn to positive considerations.

Proposition 11 *As $\iota \to 0$, the recursive monetary equilibrium allocation of the limiting economy (as $\Delta \to 0$), both with exogenous and with endogenous capital stock, converges to the efficient allocation.*

Let $\mathcal{E}_{x|y}$ denote the elasticity of variable x with respect to variable y, i.e., $\mathcal{E}_{xy} \equiv \frac{\partial x}{\partial y}\frac{y}{x}$. The following result characterizes the effect of monetary policy on real asset prices.

Proposition 12 *Consider the limiting economy (as $\Delta \to 0$). Let $z \equiv \mathcal{Z}/\varphi$, then:*

(i) If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$*,*

$$\mathcal{E}_{\varphi|\iota} = -\frac{\iota}{\iota + \frac{(1-\alpha)\theta+[\alpha+(1-\alpha)(1-\theta)][1-G(\varepsilon^*)]}{[\alpha+(1-\alpha)(1-\theta)]G(\varepsilon^*)}}$$
$$= -\frac{\iota}{\iota + \frac{\alpha-(1-\alpha)(1-\theta)z}{[\alpha+(1-\alpha)(1-\theta)]z}},$$

where ε^* *is given in part (i) of Proposition 2 .*

(ii) If $0 < \iota \leq \hat{\iota}(\lambda)$*,*

$$\mathcal{E}_{\varphi|\iota} = -\frac{\iota}{\iota + \frac{\left[\alpha+(1-\alpha)\left(1+\theta\frac{\lambda}{1-\lambda}\right)\right][1-G(\varepsilon^*)]}{G(\varepsilon^*)-(1-\alpha)\theta\frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)]}}$$
$$= -\frac{\iota}{\iota + \frac{\alpha+(1-\alpha)\left(1+\theta\frac{\lambda}{1-\lambda}\right)}{(1-\alpha)(1-\theta)\frac{\lambda}{1-\lambda}+\left[\alpha+(1-\alpha)\frac{1}{1-\lambda}\right]z}},$$

where ε^* *is given in part (ii) of Proposition 2.*

Proposition 12 provides analytical expressions for the elasticity of the asset price, φ, with respect to the policy rate, ι, both for high and low inflation regimes. In every case the elasticity is negative. In a recursive equilibrium, $\phi_t^m A_t^m = Z A_t^s y_t$, so $z \equiv \mathcal{Z}/\varphi$ as given in Proposition 2 is the value of equilibrium real money balances, $\phi_t^m A_t^m$, relative to the value of the total output, $\phi_t^s A_t^s$, (measured in terms of the dividend good). When written in terms of z, the expressions indicate that keeping the market-structure parameters α and θ constant, the impact of monetary policy on asset prices would tend to be larger in economies where aggregate real balances are a larger fraction of aggregate output.

The following corollary of Proposition 12 reports the elasticity of the real asset price to monetary policy in the limit as $\alpha \to 0$.

Corollary 5 *Consider the limiting economy (as* $\Delta \to 0$*) with* $\alpha \in [0,1]$ *and* $\lambda \in (0,1]$*. As* $\alpha \to 0$*,*

(i) If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$*,*

$$\mathcal{E}_{\varphi|\iota} \to -\frac{\iota}{\iota + \frac{\theta+(1-\theta)[1-G(\varepsilon^*)]}{(1-\theta)G(\varepsilon^*)}}$$

where ε^* *is given in part (i) of Proposition 2 .*

(ii) If $0 < \iota \leq \hat{\iota}(\lambda)$*,*

$$\mathcal{E}_{\varphi|\iota} \to -\frac{\iota}{\iota + \frac{\left(1+\theta\frac{\lambda}{1-\lambda}\right)[1-G(\varepsilon^*)]}{G(\varepsilon^*)-\theta\frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)]}} = \frac{\iota}{\iota + \frac{1-(1-\theta)\lambda}{(1-\theta)\lambda+z}},$$

where ε^* is given in part (ii) of Proposition 2.

The corollary shows that when $\hat{\imath}(\lambda) < \iota < \bar{\imath}(\lambda)$, the elasticity of the asset price with respect to monetary policy is negative and remains bounded away from zero even as z converges to zero.

To conclude, notice that in the economy with capital accumulation with production technology given by (69), the elasticity of investment with respect to ι is

$$\mathcal{E}_{\mathcal{X}|\iota} = \frac{\sigma}{1-\sigma}\mathcal{E}_{\varphi|\iota}.$$

D Proofs

D.1 Bargaining and portfolio problems

The investor's second-subperiod value function can be written as

$$W_t(\boldsymbol{a}_t, a_t^b, k_t) = \phi_t^m a_t^m + \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t \tag{99}$$

with

$$\begin{aligned}\bar{W}_t \equiv T_t + \max_{(\tilde{a}_{t+1}^m, \tilde{a}_{t+1}^s)\in\mathbb{R}_+^2} \Big[& -\phi_t^m \tilde{a}_{t+1}^m - \phi_t^s \tilde{a}_{t+1}^s \\ & + \beta\mathbb{E}_t \int V_{t+1}\left[\tilde{a}_{t+1}^m, \eta\tilde{a}_{t+1}^s + (1-\eta) A^s, \varepsilon\right] dG(\varepsilon)\Big].\end{aligned} \tag{100}$$

Proof of Lemma 1. In a nonmonetary economy, (99) reduces to

$$W_t(\boldsymbol{a}_t, a_t^b, k_t) = \phi_t^s a_t^s + a_t^b - k_t + \bar{W}_t.$$

(i) In a nonmonetary equilibrium (1) implies $\hat{a}_t^s(a_t^s, \varepsilon) = \arg\max_{0\le \hat{a}_t^s \le a_t^s} (\varepsilon y_t + \phi_t^s)\, \hat{a}_t^s$.

(ii) In a nonmonetary economy, (2) implies $\left[\overline{a}_t^s(a_t^s, \varepsilon), \overline{a}_t^b(a_t^s, \varepsilon), k_t(a_t^s, \varepsilon)\right]$ is the solution to

$$\max_{(\overline{a}_t^s, k_t)\in\mathbb{R}_+^2, \overline{a}_t^b\in\mathbb{R}} \left[(\varepsilon y_t + \phi_t^s)(\overline{a}_t^s - a_t^s) + \overline{a}_t^b - k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. } \bar{\phi}_t^s \overline{a}_t^s + \overline{a}_t^b = \bar{\phi}_t^s a_t^s \tag{101}$$

$$-\lambda\phi_t^s \overline{a}_t^s \le \overline{a}_t^b. \tag{102}$$

Notice that the first-order condition with respect to k_t implies

$$k_t(a_t^s, \varepsilon) = (1-\theta)\left\{(\varepsilon y_t + \phi_t^s)\left[\overline{a}_t^s(a_t^s, \varepsilon) - a_t^s\right] + \overline{a}_t^b(a_t^s, \varepsilon)\right\}, \tag{103}$$

so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{\overline{a}_t^s \in \mathbb{R}_+, \overline{a}_t^b \in \mathbb{R}} \left[(\varepsilon y_t + \phi_t^s) (\overline{a}_t^s - a_t^s) + \overline{a}_t^b \right] \text{ s.t. (101), and (102).}$$

Since (101) implies $\overline{a}_t^b = \bar{\phi}_t^s (a_t^s - \overline{a}_t^s)$,

$$\overline{a}_t^s (a_t^s, \varepsilon) = \arg\max_{\overline{a}_t^s} (\varepsilon - \varepsilon_t^n) \overline{a}_t^s \text{ s.t. } 0 \leq \overline{a}_t^s \text{ and } \left(\bar{\phi}_t^s - \lambda \phi_t^s \right) \overline{a}_t^s \leq \bar{\phi}_t^s a_t^s.$$

The problem has no solution (for $\varepsilon > \varepsilon_t^n$) if $\bar{\phi}_t^s - \lambda \phi_t^s \leq 0$. Provided $\bar{\phi}_t^s - \lambda \phi_t^s > 0$, the solution exists for all ε and is given by (10). Given $\overline{a}_t^s (a_t^s, \varepsilon)$, $\overline{a}_t^b (a_t^s, \varepsilon) = \bar{\phi}_t^s [a_t^s - \overline{a}_t^s (a_t^s, \varepsilon)]$ as in (11), and $k_t (a_t^s, \varepsilon)$ is given by (103), or equivalently, (12). ∎

Proof of Lemma 2.

(i) With (99), it is easy to show that the solution to the optimization problem in (1) is given by (16) and (17).

(ii) With (99), (2) can be written as

$$\max_{(\overline{a}_t^m, \overline{a}_t^s, k_t) \in \mathbb{R}_+^3, \overline{a}_t^b \in \mathbb{R}} \left\{ (\varepsilon y_t + \phi_t^s) [\overline{a}_t^s - \hat{a}_t^s(\boldsymbol{a}_t, \varepsilon)] + \phi_t^m [\overline{a}_t^m - \hat{a}_t^m(\boldsymbol{a}_t, \varepsilon)] + \overline{a}_t^b - k_t \right\}^\theta k_t^{1-\theta}$$

$$\text{s.t. } \overline{a}_t^m + p_t \overline{a}_t^s + q_t \overline{a}_t^b = a_t^m + p_t a_t^s \tag{104}$$

$$-\lambda \phi_t^s \overline{a}_t^s \leq \overline{a}_t^b. \tag{105}$$

Notice that the first-order condition with respect to k_t implies (22) so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{(\overline{a}_t^m, \overline{a}_t^s) \in \mathbb{R}_+^2, \overline{a}_t^b \in \mathbb{R}} \left\{ (\varepsilon y_t + \phi_t^s) [\overline{a}_t^s - \hat{a}_t^s(\boldsymbol{a}_t, \varepsilon)] + \phi_t^m [\overline{a}_t^m - \hat{a}_t^m(\boldsymbol{a}_t, \varepsilon)] + \overline{a}_t^b \right\}$$
$$\text{s.t. (104), and (105).}$$

Once the solution $\overline{a}_t^m(\boldsymbol{a}_t, \varepsilon)$, $\overline{a}_t^s(\boldsymbol{a}_t, \varepsilon)$, and $\overline{a}_t^b(\boldsymbol{a}_t, \varepsilon)$ to this problem has been found, $k_t (\boldsymbol{a}_t, \varepsilon)$ is given by (22). If we use (104) to substitute for $\overline{a}_t^b$, the auxiliary problem is equivalent to

$$\max_{(\overline{a}_t^m, \overline{a}_t^s) \in \mathbb{R}_+^2} \left[\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t \right) \overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t} \right) \overline{a}_t^m \right] \tag{106}$$

$$\text{s.t. } 0 \leq a_t^m + p_t a_t^s - \overline{a}_t^m - (p_t - \lambda q_t \phi_t^s)\, \overline{a}_t^s. \tag{107}$$

This problem has no solution if $p_t \leq \lambda q_t \phi_t^s$. To see this, assume $p_t \leq \lambda q_t \phi_t^s$. Set $\bar{a}_t^m = a_t^m + p_t a_t^s$ (a feasible choice), and notice (107) is satisfied by any $\bar{a}_t^s \in \mathbb{R}_+$. Thus, the value of (106) is bounded below by

$$\left(\phi_t^m - \frac{1}{q_t}\right)(a_t^m + p_t a_t^s) + \max_{\bar{a}_t^s \in \mathbb{R}_+} \left[\varepsilon y_t + (1-\lambda)\,\phi_t^s\right] \overline{a}_t^s,$$

which is arbitrarily large. Hence, condition (18) is necessary for the bargaining problem to have a solution. The Lagrangian corresponding to the auxiliary problem (106) is

$$\begin{aligned}
\mathcal{L} = &\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t\right) \overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t}\right) \overline{a}_t^m \\
&+ \xi^b \left[a_t^m + p_t a_t^s - \overline{a}_t^m - (p_t - \lambda q_t \phi_t^s)\, \overline{a}_t^s\right] + \xi^m \overline{a}_t^m + \xi^s \overline{a}_t^s,
\end{aligned}$$

where ξ^b, ξ^m, and ξ^s are the multipliers on the constraints (107), $0 \leq \overline{a}_t^m$, and $0 \leq \overline{a}_t^s$, respectively. The first-order conditions are

$$\begin{aligned}
\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t + \xi^s - (p_t - \lambda q_t \phi_t^s)\, \xi^b &= 0 \\
\phi_t^m - \frac{1}{q_t} + \xi^m - \xi^b &= 0.
\end{aligned}$$

By working out the eight possible binding patterns for the multipliers $\left(\xi^b, \xi^m, \xi^s\right)$ and collecting the optimal allocations along with the inequality restrictions implied by each case, we obtain (19)-(22). ■

D.2 Value functions

In this section we derive the value functions for brokers and investors, in a monetary economy (Lemma 3), and in a nonmonetary economy (Lemma 4).

Lemma 3 *Consider an economy with money.*

(i) The value function of a broker at the beginning of the OTC round of period t is

$$V_t^B = \Xi_t + \bar{W}_t^B, \tag{108}$$

where $\Xi_t \equiv \alpha^B \int k_t(\tilde{\boldsymbol{a}}_t, \varepsilon)\, dH_t(\tilde{\boldsymbol{a}}_t, \varepsilon)$ and $\bar{W}_t^B \equiv \beta \mathbb{E}_t V_{t+1}^B$.

(ii) The value function of an investor who enters the OTC round of period t with portfolio $\boldsymbol{a}_t$ and valuation ε is

$$V_t\left(\boldsymbol{a}_t, \varepsilon\right) = v_{It}^m\left(\varepsilon\right) a_t^m + v_{It}^s\left(\varepsilon\right) a_t^s + \bar{W}_t, \tag{109}$$

where

$$\begin{aligned} v_{It}^m\left(\varepsilon\right) \equiv\ & \phi_t^m + \left[\alpha + \left(1-\alpha\right)\left(1-\theta\right)\right] \mathbb{I}_{\{\varepsilon_t^* < \varepsilon\}} \left(\varepsilon - \varepsilon_t^*\right) y_t \frac{1}{p_t} \\ & + \left(1-\alpha\right)\theta \mathbb{I}_{\{q_t\phi_t^m < 1\}} \left(\frac{1}{q_t} - \phi_t^m\right) \\ & + \left(1-\alpha\right)\theta \mathbb{I}_{\{\varepsilon_t^{**} < \varepsilon\}} \left(\varepsilon - \varepsilon_t^{**}\right) y_t \frac{1}{p_t - \lambda q_t \phi_t^s} \\ v_{It}^s\left(\varepsilon\right) \equiv\ & \varepsilon y_t + \phi_t^s + \left[\alpha + \left(1-\alpha\right)\left(1-\theta\right)\right] \mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}} \left(\varepsilon_t^* - \varepsilon\right) y_t \\ & + \left(1-\alpha\right)\theta \left(\phi_t^m - \frac{1}{q_t}\right) \mathbb{I}_{\{1 < q_t\phi_t^m\}} \lambda q_t \phi_t^s \\ & + \left(1-\alpha\right)\theta \left(\varepsilon - \varepsilon_t^{**}\right) y_t \frac{\lambda q_t \phi_t^s - \mathbb{I}_{\{\varepsilon < \varepsilon_t^{**}\}} p_t}{p_t - \lambda q_t \phi_t^s}. \end{aligned}$$

Proof. (i) The broker's value function (108) is immediate from (4) and (6).

(ii) With (99), the value function (5) becomes

$$\begin{aligned} V_t\left(\boldsymbol{a}_t, \varepsilon\right) =\ & \bar{W}_t + \alpha\left[\left(\varepsilon y_t + \phi_t^s\right) \hat{a}_t^s(\boldsymbol{a}_t, \varepsilon) + \phi_t^m \hat{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right)\right] \\ & + \left(1-\alpha\right)\Big[\left(\varepsilon y_t + \phi_t^s\right) \bar{a}_t^s\left(\boldsymbol{a}_t, \varepsilon\right) + \phi_t^m \bar{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right) + \bar{a}_t^b\left(\boldsymbol{a}_t, \varepsilon\right) - k_t\left(\boldsymbol{a}_t, \varepsilon\right)\Big]. \end{aligned} \tag{110}$$

Substitute $k_t\left(\boldsymbol{a}_t, \varepsilon\right)$ and $\bar{a}_t^b\left(\boldsymbol{a}_t, \varepsilon\right)$ with (22) and (21), respectively, to obtain

$$\begin{aligned} V_t\left(\boldsymbol{a}_t, \varepsilon\right) =\ & \bar{W}_t + \left(\varepsilon y_t + \phi_t^s\right) a_t^s + \phi_t^m a_t^m \\ & + \left[\alpha + \left(1-\alpha\right)\left(1-\theta\right)\right] \left\{\left(\varepsilon y_t + \phi_t^s\right)\left[\hat{a}_t^s(\boldsymbol{a}_t, \varepsilon) - a_t^s\right] + \phi_t^m\left[\hat{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right) - a_t^m\right]\right\} \\ & + \left(1-\alpha\right)\theta\left\{\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t} p_t\right)\left[\bar{a}_t^s\left(\boldsymbol{a}_t, \varepsilon\right) - a_t^s\right] + \left(\phi_t^m - \frac{1}{q_t}\right)\left[\bar{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right) - a_t^m\right]\right\}. \end{aligned}$$

Then use Lemma 2 to replace the post-trade allocations $\hat{a}_t^s(\boldsymbol{a}_t, \varepsilon)$, $\hat{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right)$, $\bar{a}_t^s\left(\boldsymbol{a}_t, \varepsilon\right)$, and $\bar{a}_t^m\left(\boldsymbol{a}_t, \varepsilon\right)$, and rearrange terms to arrive at (109). ■

Lemma 4 *Consider an economy without money.*

(i) The value function of a broker at the beginning of the OTC round of period t is

$$V_t^B = \Xi_t + \bar{W}_t^B, \tag{111}$$

where $\Xi_t \equiv \alpha^B \int k_t\left(\tilde{a}_t^s, \varepsilon\right) dH_t\left(\tilde{a}_t^s, \varepsilon\right)$ and $\bar{W}_t^B \equiv \beta \mathbb{E}_t V_{t+1}^B$.

(ii) The value function of an investor who enters the OTC round of period t with equity holding a_t^s and valuation ε is

$$V_t\left(a_t^s,\varepsilon\right)=\left\{\varepsilon y_t+\phi_t^s+(1-\alpha)\,\theta\left(\varepsilon-\varepsilon_t^n\right)y_t\left[\chi\left(\varepsilon_t^n,\varepsilon\right)\frac{\bar{\phi}_t^s}{\bar{\phi}_t^s-\lambda\phi_t^s}-1\right]\right\}a_t^s+\bar{W}_t, \tag{112}$$

where

$$\bar{W}_t\equiv\max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left[-\phi_t^s\tilde{a}_{t+1}^s+\beta\mathbb{E}_t\int V_{t+1}\left[\eta\tilde{a}_{t+1}^s+(1-\eta)\,A^s,\varepsilon\right]dG(\varepsilon)\right]. \tag{113}$$

Proof. (i) The broker's value function (111) is immediate from (4) and (6) under the assumption that investors carry no money.

(ii) In a nonmonetary economy, (99) reduces to

$$W_t(a_t^s,a_t^b,k_t)=\phi_t^s a_t^s+a_t^b-k_t+\bar{W}_t, \tag{114}$$

where $\bar{W}_t$ is given by (113). With (114) and Lemma 1, (5) reduces to (112). ■

D.3 Euler equations

In this section we derive the Euler equations that characterize the optimal portfolio choices in the second subperiod, in a monetary economy (Lemma 5) and in a nonmonetary economy (Lemma 6).

Lemma 5 *Consider an economy with money. Let $\left(\tilde{a}_{It+1}^m,\tilde{a}_{It+1}^s\right)$ denote an individual investor's portfolio choice in the second subperiod of period t. The portfolio $\left(\tilde{a}_{It+1}^m,\tilde{a}_{It+1}^s\right)$ is optimal if and only if it satisfies*

$$\left(\phi_t^m-\beta\mathbb{E}_t\bar{v}_{It+1}^m\right)\tilde{a}_{It+1}^m=0\leq\phi_t^m-\beta\mathbb{E}_t\bar{v}_{It+1}^m \tag{115}$$

$$\left(\phi_t^s-\beta\eta\mathbb{E}_t\bar{v}_{It+1}^s\right)\tilde{a}_{It+1}^s=0\leq\phi_t^s-\beta\eta\mathbb{E}_t\bar{v}_{It+1}^s, \tag{116}$$

where

$$\begin{aligned}\bar{v}_{It+1}^m\equiv\;&\phi_{t+1}^m+(1-\alpha)\,\theta\left(\frac{1}{q_{t+1}}-\phi_{t+1}^m\right)\mathbb{I}_{\left\{q_{t+1}\phi_{t+1}^m<1\right\}}\\&+\left[\alpha+(1-\alpha)\,(1-\theta)\right]\int_{\varepsilon_{t+1}^*}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{t+1}^*\right)y_{t+1}\frac{1}{p_{t+1}}dG(\varepsilon)\\&+(1-\alpha)\,\theta\frac{1}{p_{t+1}-\lambda q_{t+1}\phi_{t+1}^s}\int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{t+1}^{**}\right)y_{t+1}dG(\varepsilon),\end{aligned}$$

and

$$
\begin{aligned}
\bar{v}^s_{It+1} \equiv\; & \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)\,\theta \left(\phi^m_{t+1} - \frac{1}{q_{t+1}}\right) \mathbb{I}_{\left\{1<q_{t+1}\phi^m_{t+1}\right\}} \lambda q_{t+1}\phi^s_{t+1} \\
& + \left[\alpha + (1-\alpha)(1-\theta)\right] \int_{\varepsilon_L}^{\varepsilon^*_{t+1}} \left(\varepsilon^*_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
& + (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^{**}_{t+1}} \left(\varepsilon^{**}_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1}\phi^s_{t+1}}{p_{t+1} - \lambda q_{t+1}\phi^s_{t+1}} \int_{\varepsilon^{**}_{t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^{**}_{t+1}\right) y_{t+1} dG(\varepsilon)\right].
\end{aligned}
$$

Proof. With (109) and (100), the portfolio problem of an investor in the second subperiod can be written as

$$
\begin{aligned}
\bar{W}_t \equiv\; & T_t + \beta \mathbb{E}_t \left[\bar{W}_{t+1} + \bar{v}^s_{It+1}(1-\eta) A^s\right] \\
& + \max_{(\tilde{a}^m_{t+1}, \tilde{a}^s_{t+1}) \in \mathbb{R}^2_+} \left[\left(\beta \mathbb{E}_t \bar{v}^m_{It+1} - \phi^m_t\right) \tilde{a}^m_{t+1} + \left(\beta\eta \mathbb{E}_t \bar{v}^s_{It+1} - \phi^s_t\right) \tilde{a}^s_{t+1}\right],
\end{aligned}
$$

where $\bar{v}^k_{It+1} \equiv \int v^k_{It+1}(\varepsilon)\, dG(\varepsilon)$ for $k \in \{m, s\}$. ■

Lemma 6 *Consider an economy with no money. Let $\tilde{a}^s_{It+1}$ denote equity holding chosen by an individual investor in the second subperiod of period t. Then $\tilde{a}^s_{It+1}$ is optimal if and only if it satisfies*

$$
\begin{aligned}
& -\phi^s_t + \beta\eta \mathbb{E}_t \Bigg\{ \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)\,\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}} \left(\varepsilon^n_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
& + \frac{\lambda \phi^s_{t+1}}{\bar{\phi}^s_{t+1} - \lambda \phi^s_{t+1}} \int_{\varepsilon^n_{t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n_{t+1}\right) y_{t+1} dG(\varepsilon)\Bigg]\Bigg\} \\
& \leq 0, \text{ with "} = \text{" if } \tilde{a}^s_{It+1} > 0.
\end{aligned} \tag{117}
$$

Proof. With (112) and (113), the portfolio problem of an investor in the second subperiod can be written as

$$
\begin{aligned}
& \max_{\tilde{a}^s_{t+1} \in \mathbb{R}_+} \Bigg[-\phi^s_t + \beta\eta \mathbb{E}_t \Bigg\{ \bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)\,\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}} \left(\varepsilon^n_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
& + \frac{\lambda \phi^s_{t+1}}{\bar{\phi}^s_{t+1} - \lambda \phi^s_{t+1}} \int_{\varepsilon^n_{t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n_{t+1}\right) y_{t+1} dG(\varepsilon)\Bigg]\Bigg\}\Bigg] \tilde{a}^s_{t+1}.
\end{aligned}
$$

■

D.4 Market-clearing conditions

In this section we derive the market-clearing conditions for equity and bonds in the OTC round, in a monetary economy (Lemma 7) and in a nonmonetary economy (Lemma 8).

Lemma 7 *In a monetary equilibrium, the market-clearing conditions for equity,* $\hat{A}^s_{It}+\bar{A}^s_{It}=A^s$*, and bonds,* $\bar{A}^b_{It}=0$*, in the OTC round are:*

$$0=\alpha\left[1-G\left(\varepsilon^*_t\right)\right]\frac{A^m_t+p_tA^s}{p_t}+(1-\alpha)\left[1-G\left(\varepsilon^{**}_t\right)\right]\frac{A^m_t+p_tA^s}{p_t-\lambda q_t\phi^s_t}-A^s \tag{118}$$

$$\begin{aligned}0=(1-\alpha)\Bigg\{&\left\{1-\mathbb{I}_{\{1<q_t\phi^m_t\}}-\mathbb{I}_{\{q_t\phi^m_t=1\}}\left[1-\chi\left(1,q_t\phi^m_t\right)\right]\right\}G\left(\varepsilon^{**}_t\right)\\&-\frac{\lambda q_t\phi^s_t}{p_t-\lambda q_t\phi^s_t}\left[1-G\left(\varepsilon^{**}_t\right)\right]\Bigg\}\frac{A^m_t+p_tA^s}{q_t}.\end{aligned} \tag{119}$$

Proof. By Lemma 2, the investors' aggregate post-trade holdings of equity in the OTC round of period t are

$$\begin{aligned}\bar{A}^s_{It}&=(1-\alpha)N_I\int\overline{a}^s_t(\boldsymbol{a}_t,\varepsilon)dH_t(\boldsymbol{a}_t,\varepsilon)=(1-\alpha)\left[1-G\left(\varepsilon^{**}_t\right)\right]\frac{A^m_t+p_tA^s}{p_t-\lambda q_t\phi^s_t}\\\hat{A}^s_{It}&=\alpha N_I\int\hat{a}^s_t(\boldsymbol{a}_t,\varepsilon)dH_t(\boldsymbol{a}_t,\varepsilon)=\alpha\left[1-G\left(\varepsilon^*_t\right)\right]\frac{A^m_t+p_tA^s}{p_t}\end{aligned}$$

and the the investors' aggregate post-trade holdings of bonds in the OTC round of period t are

$$\begin{aligned}\bar{A}^b_{It}&=(1-\alpha)N_I\int\overline{a}^b_t(\boldsymbol{a}_t,\varepsilon)dH_t(\boldsymbol{a}_t,\varepsilon)\\&=(1-\alpha)\Bigg\{\left\{1-\mathbb{I}_{\{1<q_t\phi^m_t\}}-\mathbb{I}_{\{q_t\phi^m_t=1\}}\left[1-\chi\left(1,q_t\phi^m_t\right)\right]\right\}G\left(\varepsilon^{**}_t\right)\\&\quad-\frac{\lambda q_t\phi^s_t}{p_t-\lambda q_t\phi^s_t}\left[1-G\left(\varepsilon^{**}_t\right)\right]\Bigg\}\frac{A^m_t+p_tA^s}{q_t}.\end{aligned}$$

■

Lemma 8 *In a nonmonetary equilibrium, the market-clearing condition for equity* $\hat{A}^s_{It}+\bar{A}^s_{It}=A^s$ *(or bonds,* $\bar{A}^b_{It}=0$*) in the OTC round is:*

$$1=\left[1-G\left(\varepsilon^n_t\right)\right]\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}. \tag{120}$$

Proof. By Lemma 1, the investors' aggregate post-trade holdings of equity in the OTC round of period t are

$$\bar{A}^s_{It} = (1-\alpha)\,N_I \int \overline{a}^s_t(a_t,\varepsilon)dH_t(a_t,\varepsilon) = (1-\alpha)\int \chi\left(\varepsilon^n_t,\varepsilon\right)\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}A^s dG\left(\varepsilon\right)$$

$$\hat{A}^s_{It} = \alpha N_I \int \hat{a}^s_t(a_t,\varepsilon)dH_t(a_t,\varepsilon) = \alpha A^s$$

and the the aggregate post-trade holdings of bonds for agents who trade in the bond market in the OTC round of period t are

$$\bar{A}^b_{It} = (1-\alpha)\,N_I \int \overline{a}^b_t(a_t,\varepsilon)dH_t(a_t,\varepsilon) = (1-\alpha)\int \bar{\phi}^s_t\left[1-\chi\left(\varepsilon^n_t,\varepsilon\right)\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t}\right]A^s dG\left(\varepsilon\right).$$

■

D.5 Equilibrium conditions

In this section we state the operational definitions of monetary and nonmonetary equilibrium that are used in the analysis.

D.5.1 Sequential nonmonetary equilibrium

Definition 4 *A (sequential) nonmonetary equilibrium is an allocation* $\{\tilde{a}^s_{It+1}\}^\infty_{t=0}$ *and a sequence of prices,* $\{\phi^s_t,\bar{\phi}^s_t\}^\infty_{t=0}$*, that satisfy the portfolio-optimality condition, (117) (with* $\tilde{a}^k_{It+1}=\tilde{A}^k_{It+1}$*), and the market-clearing conditions* $\tilde{A}^s_{It+1}=A^s$ *and (120).*

Definition 4 follows from Definition 1 after recognizing that all investors choose the same end-of-period portfolio that is characterized by the Euler equations derived in Lemma 6, and using the explicit version of the market clearing condition for equity and bonds in the OTC round derived in Lemma 8. Given the equilibrium objects in Definition 4, the bargaining outcomes, which are part of Definition 1 but not Definition 4, are immediate from Lemma 1.

According to Definition 4, a nonmonetary equilibrium can be characterized by sequence of prices, $\{\phi^s_t,\bar{\phi}^s_t\}^\infty_{t=0}$ and an allocation $\left\{\tilde{A}^s_{It+1}\right\}^\infty_{t=0}$ that satisfy the market-clearing conditions

$$A^s = \tilde{A}^s_{It+1} \tag{121}$$

$$1 = \left[1-G\left(\varepsilon^n_t\right)\right]\frac{\bar{\phi}^s_t}{\bar{\phi}^s_t-\lambda\phi^s_t} \tag{122}$$

and the portfolio-optimality condition

$$
\begin{aligned}
&-\phi_t^s+\beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon}y_{t+1}+\phi_{t+1}^s+(1-\alpha)\,\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon_{t+1}^n}\left(\varepsilon_{t+1}^n-\varepsilon\right)y_{t+1}dG(\varepsilon)\\
&+\frac{\lambda\phi_{t+1}^s}{\bar{\phi}_{t+1}^s-\lambda\phi_{t+1}^s}\int_{\varepsilon_{t+1}^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon_{t+1}^n\right)y_{t+1}dG(\varepsilon)\Bigg]\Bigg\}\\
&\leq 0,\text{ with} “=” \text{ if } \tilde{A}_{It+1}^s>0,
\end{aligned}
\tag{123}
$$

where ε_t^n is given by (8).

D.5.2 Recursive nonmonetary equilibrium

The following result summarizes the conditions that characterize a recursive nonmonetary equilibrium (RNE).

Lemma 9 *A recursive nonmonetary equilibrium is a vector* $(\varepsilon^n,\phi^s,\tilde{A}_I^s)$ *that satisfies the following conditions*

$$
\begin{aligned}
0&=\tilde{A}_I^s-A^s\\
1&=\left[1-G\left(\varepsilon^n\right)\right]\frac{\varepsilon^n+\phi^s}{\varepsilon^n+(1-\lambda)\,\phi^s}\\
\phi^s&\geq\bar{\beta}\eta\left\{\bar{\varepsilon}+\phi^s+(1-\alpha)\,\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG(\varepsilon)+\frac{\lambda\phi^s}{\varepsilon^n+(1-\lambda)\,\phi^s}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG(\varepsilon)\right]\right\}
\end{aligned}
$$

with “=” if $\tilde{A}_I^s>0$.

Proof. The equilibrium conditions in the statement of the lemma are obtained from (121)-(123) by using $\phi_t^s=\phi^s y_t$, $\bar{\phi}_t^s=\bar{\phi}^s y_t$, $\tilde{A}_{It}^s=\tilde{A}_I^s$, and $\varepsilon_t^n=\left(\bar{\phi}_t^s-\phi_t^s\right)\frac{1}{y_t}=\bar{\phi}^s-\phi^s\equiv\varepsilon^n$. ■

The first equation in the statement of Lemma 9 is the second-subperiod market-clearing condition for equity. The second equation is the first-subperiod market-clearing condition for equity (or bonds). The third condition is the investor's Euler equation for equity.

D.5.3 Sequential monetary equilibrium

Definition 5 *A (sequential) monetary equilibrium is an allocation* $\left\{(\tilde{a}_{It+1}^k)_{k\in\{m,s\}}\right\}_{t=0}^{\infty}$ *and a sequence of prices,* $\{p_t,q_t,\phi_t^m,\phi_t^s\}_{t=0}^{\infty}$*, that satisfy the two optimality conditions, (115) and (116) (with* $\tilde{a}_{It+1}^k=\tilde{A}_{It+1}^k$*), and the four market-clearing conditions,* $\tilde{A}_{It+1}^s=A^s$*,* $\tilde{A}_{It+1}^m=A_{t+1}^m$*, (118), and (119).*

Definition 5 follows from Definition 1 after recognizing that all investors choose the same end-of-period portfolio that is characterized by the Euler equation derived in Lemma 5, and using the explicit version of the market clearing condition for equity in the OTC round derived in Lemma 7. Given the equilibrium objects in Definition 5, the bargaining outcomes, which are part of Definition 1 but not Definition 5, are immediate from Lemma 2.

According to Definition 5, a monetary equilibrium can be characterized by sequence of prices, $\{p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$ and an allocation $\{(\tilde{A}_{It+1}^k)_{k\in\{m,s\}}\}_{t=0}^{\infty}$ that satisfy the following market-clearing conditions

$$
\begin{aligned}
&0 = \tilde{A}_{It+1}^s - A^s \\
&0 = \tilde{A}_{It+1}^m - A_{t+1}^m \\
&0 = \alpha\left[1 - G\left(\varepsilon_t^*\right)\right]\frac{A_t^m + p_t A^s}{p_t} + (1-\alpha)\left[1 - G\left(\varepsilon_t^{**}\right)\right]\frac{A_t^m + p_t A^s}{p_t - \lambda q_t \phi_t^s} - A^s \\
&0 = (1-\alpha)\left\{\left[1 - \mathbb{I}_{\{1<q_t\phi_t^m\}} - \mathbb{I}_{\{q_t\phi_t^m=1\}}\left(1-\chi_{11}\right)\right]G\left(\varepsilon_t^{**}\right) - \frac{\lambda q_t \phi_t^s}{p_t - \lambda q_t \phi_t^s}\left[1 - G\left(\varepsilon_t^{**}\right)\right]\right\}\frac{A_t^m + p_t A^s}{q_t}
\end{aligned}
$$

and optimality conditions

$$
\begin{aligned}
\left(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m\right)\tilde{A}_{It+1}^m &= 0 \le \phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m \\
\left(\phi_t^s - \beta\eta \mathbb{E}_t \bar{v}_{It+1}^s\right)\tilde{A}_{It+1}^s &= 0 \le \phi_t^s - \beta\eta \mathbb{E}_t \bar{v}_{It+1}^s,
\end{aligned}
$$

where ε_t^* is given by (14), ε_t^{**} is given by (15), $i_t^m \equiv \frac{1}{q_t\phi_t^m} - 1$, and

$$
\begin{aligned}
\bar{v}_{It+1}^m \equiv{}& \phi_{t+1}^m + (1-\alpha)\,\theta\left(\frac{1}{q_{t+1}} - \phi_{t+1}^m\right)\mathbb{I}_{\{q_{t+1}\phi_{t+1}^m<1\}} \\
&+ \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_{t+1}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^*\right)y_{t+1}\frac{1}{p_{t+1}}dG(\varepsilon) \\
&+ (1-\alpha)\,\theta\frac{1}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^s}\int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^{**}\right)y_{t+1}dG(\varepsilon) \\
\bar{v}_{It+1}^s \equiv{}& \bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + (1-\alpha)\,\theta\left(\phi_{t+1}^m - \frac{1}{q_{t+1}}\right)\mathbb{I}_{\{1<q_{t+1}\phi_{t+1}^m\}}\lambda q_t \phi_{t+1}^s \\
&+ \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_L}^{\varepsilon_{t+1}^*}\left(\varepsilon_{t+1}^* - \varepsilon\right)y_{t+1}dG\left(\varepsilon\right) \\
&+ (1-\alpha)\,\theta\left[\int_{\varepsilon_L}^{\varepsilon_{t+1}^{**}}\left(\varepsilon_{t+1}^{**} - \varepsilon\right)y_{t+1}dG\left(\varepsilon\right) + \frac{\lambda q_{t+1}\phi_{t+1}^s}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^s}\int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^{**}\right)y_{t+1}dG\left(\varepsilon\right)\right].
\end{aligned}
$$

D.5.4 Sequential monetary equilibrium with credit

The following result states that the credit market would be inactive if the net nominal interest rate on bonds, $i_t^m \equiv \frac{1}{q_t \phi_t^m} - 1$, were negative.

Lemma 10 *Consider a monetary equilibrium. If the bond market is active in period t, then $q_t \phi_t^m \leq 1$.*

Proof. In an equilibrium with $1 < q_t \phi_t^m$ the market-clearing condition (119) becomes

$$0 = (1-\alpha) \frac{\lambda \phi_t^s}{p_t - \lambda q_t \phi_t^s} \left[1 - G\left(\varepsilon_t^{**}\right)\right] \left(A_t^m + p_t A^s\right).$$

This condition can only hold if $\left[1 - G\left(\varepsilon_t^{**}\right)\right] \left(A_t^m + p_t A^s\right) = 0$, i.e., if the bond market is inactive. The condition $1 < q_t \phi_t^m$ implies bond demand is nil, so the bond market can only clear with no trade. ■

According to Lemma 10, a monetary equilibrium with an active bond market can be characterized by sequence of prices, $\{p_t, q_t, \phi_t^m, \phi_t^s\}_{t=0}^{\infty}$ and an allocation $\{(\tilde{A}_{It+1}^k)_{k \in \{m,s\}}\}_{t=0}^{\infty}$ that satisfy the following market-clearing conditions

$$\begin{aligned}
&0 = \tilde{A}_{It+1}^s - A^s \\
&0 = \tilde{A}_{It+1}^m - A_{t+1}^m \\
&0 = \alpha \left[1 - G\left(\varepsilon_t^*\right)\right] \frac{A_t^m + p_t A^s}{p_t} + (1-\alpha) \left[1 - G\left(\varepsilon_t^{**}\right)\right] \frac{A_t^m + p_t A^s}{p_t - \lambda q_t \phi_t^s} - A^s \\
&0 = (1-\alpha) \left\{ \left[1 - \mathbb{I}_{\{q_t \phi_t^m = 1\}} \left(1 - \chi_{11}\right)\right] G\left(\varepsilon_t^{**}\right) - \frac{\lambda q_t \phi_t^s}{p_t - \lambda q_t \phi_t^s} \left[1 - G\left(\varepsilon_t^{**}\right)\right] \right\} \frac{A_t^m + p_t A^s}{q_t}
\end{aligned}$$

and optimality conditions

$$\begin{aligned}
\left(\phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m\right) \tilde{A}_{It+1}^m = 0 \leq \phi_t^m - \beta \mathbb{E}_t \bar{v}_{It+1}^m \\
\left(\phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{It+1}^s\right) \tilde{A}_{It+1}^s = 0 \leq \phi_t^s - \beta \eta \mathbb{E}_t \bar{v}_{It+1}^s,
\end{aligned}$$

where

$$\bar{v}_{It+1}^{m} \equiv \phi_{t+1}^{m} + (1-\alpha)\,\theta\left(\frac{1}{q_{t+1}} - \phi_{t+1}^{m}\right)$$
$$+ [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_{t+1}^{*}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^{*}\right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \frac{1}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^{s}} \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^{**}\right) y_{t+1} dG(\varepsilon)$$
$$\bar{v}_{It+1}^{s} \equiv \bar{\varepsilon} y_{t+1} + \phi_{t+1}^{s}$$
$$+ [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{t+1}^{*}} \left(\varepsilon_{t+1}^{*} - \varepsilon\right) y_{t+1} dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon_{t+1}^{**}} \left(\varepsilon_{t+1}^{**} - \varepsilon\right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1}\phi_{t+1}^{s}}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^{s}} \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^{**}\right) y_{t+1} dG(\varepsilon)\right].$$

D.5.5 Recursive monetary equilibrium with credit

The following result summarizes the conditions that characterize a recursive monetary equilibrium (RME).

Lemma 11 *A recursive monetary equilibrium (with credit) is a vector $(\varepsilon^{*}, \varepsilon^{**}, \phi^{s}, Z)$ that satisfies*

$$0 = \left\{\alpha\left[1 - G(\varepsilon^{*})\right] + (1-\alpha)\left[1 - G(\varepsilon^{**})\right] \frac{\varepsilon^{**} + \phi^{s}}{\varepsilon^{**} + (1-\lambda)\phi^{s}}\right\}\left(\frac{Z}{\varepsilon^{*} + \phi^{s}} + 1\right) - 1$$
$$0 = (1-\alpha)\left\{G(\varepsilon^{**})\left[1 - \mathbb{I}_{\{\varepsilon^{*}=\varepsilon^{**}\}}(1-\chi_{11})\right] - \left[1 - G(\varepsilon^{**})\right] \frac{\lambda\phi^{s}}{\varepsilon^{**} + (1-\lambda)\phi^{s}}\right\}(Z + \varepsilon^{*} + \phi^{s})$$

where $\chi_{11} \in [0,1]$, *and*

$$i^{p} = (1-\alpha)\,\theta\left(\frac{\varepsilon^{**} + \phi^{s}}{\varepsilon^{*} + \phi^{s}} - 1\right) + [\alpha + (1-\alpha)(1-\theta)] \frac{1}{\varepsilon^{*} + \phi^{s}} \int_{\varepsilon^{*}}^{\varepsilon_H} (\varepsilon - \varepsilon^{*})\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \frac{\varepsilon^{**} + \phi^{s}}{\varepsilon^{*} + \phi^{s}} \frac{1}{\varepsilon^{**} + (1-\lambda)\phi^{s}} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)$$
$$\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta}\phi^{s} = \bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^{*}} (\varepsilon^{*} - \varepsilon)\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta\left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) + \frac{\lambda\phi^{s}}{\varepsilon^{**} + (1-\lambda)\phi^{s}} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right].$$

Proof. The equilibrium conditions in the statement of the lemma are obtained from the ones in Section D.5.4 by using $\phi_t^{s} = \phi^{s} y_t$, $p_t \phi_t^{m} \equiv \bar{\phi}_{mt}^{s} = \bar{\phi}_{m}^{s} y_t$, $p_t/q_t \equiv \bar{\phi}_{bt}^{s} = \bar{\phi}_{b}^{s} y_t$, $\phi_t^{m} A_t^{m} = Z A^{s} y_t$,

$\varepsilon_t^* = (p_t\phi_t^m - \phi_t^s)\frac{1}{y_t} = \bar{\phi}_m^s - \phi^s \equiv \varepsilon^*$, $\varepsilon_t^{**} = (p_t/q_t - \phi_t^s)\frac{1}{y_t} = \bar{\phi}_b^s - \phi^s \equiv \varepsilon^{**}$, $p_t = \frac{(\varepsilon^*+\phi^s)A_t^m}{ZA^s}$, $\phi_t^m = \frac{ZA^s y_t}{A_t^m}$, $q_t = \frac{(\varepsilon^*+\phi^s)A_t^m}{(\varepsilon^{**}+\phi^s)ZA^s y_t}$, $\phi_{t+1}^s/\phi_t^s = \bar{\phi}_{mt+1}^s/\bar{\phi}_{mt}^s = \bar{\phi}_{bt+1}^s/\bar{\phi}_{bt}^s = \gamma_{t+1}$, $p_{t+1}/p_t = \mu$, and $\phi_t^m/\phi_{t+1}^m = q_{t+1}/q_t = \mu/\gamma_{t+1}$. ■

The first and second equations in Lemma 11 are the first-subperiod market-clearing condition for equity and bonds, respectively. The remaining two conditions are the investor's Euler equations for money and equity, respectively.

D.6 Continuous-time limiting economy

In this section we derive the equilibrium conditions for the continuous-time limiting economy.

D.6.1 Equilibrium conditions

Lemma 12 *Consider the limiting economy (as $\Delta \to 0$). A recursive nonmonetary equilibrium is a pair (ε^n, φ) that satisfies*

$$1 = \frac{1 - G(\varepsilon^n)}{1 - \lambda}$$

$$\varphi = \bar{\varepsilon} + (1-\alpha)\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\right].$$

Proof. From Lemma 9, if the period length is Δ, an equilibrium is a pair $(\varepsilon^n, \Phi^s(\Delta))$ that satisfies

$$1 = [1 - G(\varepsilon^n)]\frac{\varepsilon^n + \Phi^s(\Delta)}{\varepsilon^n + (1-\lambda)\Phi^s(\Delta)}$$

$$\Phi^s(\Delta) = \bar{\beta}\eta\Bigg\{\bar{\varepsilon} + \Phi^s(\Delta) + (1-\alpha)\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda\Phi^s(\Delta)}{\varepsilon^n + (1-\lambda)\Phi^s(\Delta)}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\Bigg]\Bigg\}$$

This can be written as

$$1 = [1 - G(\varepsilon^n)]\frac{\varepsilon^n\Delta + \Phi^s(\Delta)\Delta}{\varepsilon^n\Delta + (1-\lambda)\Phi^s(\Delta)\Delta}$$

$$\frac{r + \delta - g + g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\Delta = \bar{\varepsilon} + (1-\alpha)\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon) + \frac{\lambda\Phi^s(\Delta)\Delta}{\varepsilon^n\Delta + (1-\lambda)\Phi^s(\Delta)\Delta}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon)\Bigg]$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ■

Lemma 13 *Consider the limiting economy (as $\Delta \to 0$). A recursive monetary equilibrium (with credit) is a vector $(\varepsilon^{*}, \varepsilon^{**}, \varphi, \mathcal{Z})$ that satisfies*

$$0 = \left\{ \alpha \left[1 - G\left(\varepsilon^{*}\right) \right] + (1-\alpha) \left[1 - G\left(\varepsilon^{**}\right) \right] \frac{1}{1-\lambda} \right\} \left(\frac{\mathcal{Z}}{\varphi} + 1 \right) - 1 \tag{124}$$

$$0 = G\left(\varepsilon^{**}\right) \left[1 - \mathbb{I}_{\{\varepsilon^{*}=\varepsilon^{**}\}} \left(1-\chi_{11}\right) \right] - \left[1 - G\left(\varepsilon^{**}\right) \right] \frac{\lambda}{1-\lambda} \tag{125}$$

where $\chi_{11} \in [0,1]$, and

$$\begin{aligned} \iota\varphi = {} & (1-\alpha)\,\theta\left(\varepsilon^{**} - \varepsilon^{*}\right) + \left[\alpha + (1-\alpha)(1-\theta)\right] \int_{\varepsilon^{*}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^{*}\right) dG(\varepsilon) \\ & + (1-\alpha)\,\theta \frac{1}{1-\lambda} \int_{\varepsilon^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^{**}\right) dG(\varepsilon) \end{aligned} \tag{126}$$

$$\begin{aligned} \varphi = {} & \bar{\varepsilon} + \left[\alpha + (1-\alpha)(1-\theta)\right] \int_{\varepsilon_L}^{\varepsilon^{*}} \left(\varepsilon^{*} - \varepsilon\right) dG\left(\varepsilon\right) \\ & + (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^{**}} \left(\varepsilon^{**} - \varepsilon\right) dG\left(\varepsilon\right) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^{**}\right) dG\left(\varepsilon\right) \right]. \end{aligned} \tag{127}$$

Proof. If the period length is Δ, the equilibrium conditions in Lemma 11 generalize to

$$0 = \left\{ \alpha \left[1 - G\left(\varepsilon^{*}\right) \right] + (1-\alpha) \left[1 - G\left(\varepsilon^{**}\right) \right] \frac{\varepsilon^{**} + \Phi^{s}\left(\Delta\right)}{\varepsilon^{**} + (1-\lambda)\,\Phi^{s}\left(\Delta\right)} \right\} \left(\frac{\mathrm{Z}\left(\Delta\right)}{\varepsilon^{*} + \Phi^{s}\left(\Delta\right)} + 1 \right) - 1$$

$$\begin{aligned} 0 = (1-\alpha) \Bigg\{ & G\left(\varepsilon^{**}\right) \left[1 - \mathbb{I}_{\{\varepsilon^{*}=\varepsilon^{**}\}} \left(1-\chi_{11}\right) \right] \\ & - \left[1 - G\left(\varepsilon^{**}\right) \right] \frac{\lambda \Phi^{s}\left(\Delta\right)}{\varepsilon^{**} + (1-\lambda)\,\Phi^{s}\left(\Delta\right)} \Bigg\} \left[\mathrm{Z}\left(\Delta\right) + \varepsilon^{*} + \Phi^{s}\left(\Delta\right) \right] \end{aligned}$$

where $\chi_{11} \in [0,1]$, and

$$
\begin{aligned}
i^p &= (1-\alpha)\,\theta \frac{\varepsilon^{**}-\varepsilon^*}{\varepsilon^*+\Phi^s(\Delta)} \\
&+ [\alpha + (1-\alpha)(1-\theta)] \frac{1}{\varepsilon^*+\Phi^s(\Delta)} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
&+ (1-\alpha)\,\theta \frac{\varepsilon^{**}+\Phi^s(\Delta)}{\varepsilon^*+\Phi^s(\Delta)} \frac{1}{\varepsilon^{**}+(1-\lambda)\Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
\frac{r+\delta-g+g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\,\Delta &= \bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\
&+ (1-\alpha)\,\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) \\
&+ \frac{\lambda \Phi^s(\Delta)}{\varepsilon^{**}+(1-\lambda)\Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon) \Bigg].
\end{aligned}
$$

These conditions can be rewritten as

$$
0 = \left\{ \alpha [1 - G(\varepsilon^*)] + (1-\alpha)[1 - G(\varepsilon^{**})] \frac{\varepsilon^{**}\Delta + \Phi^s(\Delta)\,\Delta}{\varepsilon^{**}\Delta + (1-\lambda)\Phi^s(\Delta)\,\Delta} \right\} \left(\frac{\mathrm{Z}(\Delta)\,\Delta}{\varepsilon^*\Delta + \Phi^s(\Delta)\,\Delta} + 1 \right) - 1
$$

$$
\begin{aligned}
0 = (1-\alpha) \Bigg\{ & G(\varepsilon^{**}) \left[1 - \mathbb{I}_{\{\varepsilon^*=\varepsilon^{**}\}}(1-\chi_{11})\right] \\
& - [1 - G(\varepsilon^{**})] \frac{\lambda \Phi^s(\Delta)\,\Delta}{\varepsilon^{**}\Delta + (1-\lambda)\Phi^s(\Delta)\,\Delta} \Bigg\} [\mathrm{Z}(\Delta)\,\Delta + \varepsilon^*\Delta + \Phi^s(\Delta)\,\Delta]
\end{aligned}
$$

where $\chi_{11} \in [0,1]$, and

$$
\begin{aligned}
\frac{i^p}{\Delta} &= (1-\alpha)\,\theta \frac{\varepsilon^{**}-\varepsilon^*}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta} \\
&+ [\alpha + (1-\alpha)(1-\theta)] \frac{1}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
&+ (1-\alpha)\,\theta \frac{\varepsilon^{**}\Delta+\Phi^s(\Delta)\,\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta} \frac{1}{\varepsilon^{**}\Delta+(1-\lambda)\Phi^s(\Delta)\,\Delta} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
\frac{r+\delta-g+g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\,\Delta &= \bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\
&+ (1-\alpha)\,\theta \Bigg[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) \\
&+ \frac{\lambda \Phi^s(\Delta)\,\Delta}{\varepsilon^{**}\Delta+(1-\lambda)\Phi^s(\Delta)\,\Delta} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon) \Bigg].
\end{aligned}
$$

Take the limit as $\Delta \to 0$ to arrive at the conditions in the statement of the lemma. ∎

D.6.2 Existence of equilibrium

Proof of Proposition 1. The conditions (32) and (33) in the statement of the proposition are the equilibrium conditions derived in Lemma 12. Clearly for any $\lambda \in [0,1]$ there is a unique ε^n that satisfies (33), and given ε^n, the normalized equity price φ is given by (32). ■

Lemma 14 *In a RNE,*

$$\begin{aligned}\frac{d\varepsilon^n}{d\lambda} &= \frac{1}{G'(\varepsilon^n)} > 0 \\ \frac{d\varphi^n}{d\lambda} &= (1-\alpha)\,\theta\frac{1}{(1-\lambda)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon) > 0.\end{aligned}$$

Proof. The first result is obtained by implicitly differentiating (33). For the second result, differentiate (32):

$$\begin{aligned}\frac{d}{d\lambda}\varphi^n &= (1-\alpha)\,\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right] \\ &= (1-\alpha)\,\theta\left[G(\varepsilon^n)\frac{d\varepsilon^n}{d\lambda}-\frac{\lambda}{1-\lambda}\left[1-G(\varepsilon^n)\right]\frac{d\varepsilon^n}{d\lambda}+\frac{1}{(1-\lambda)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right] \\ &= (1-\alpha)\,\theta\frac{1}{(1-\lambda)^2}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon).\end{aligned}$$

■

Proof of Proposition 2. The equilibrium conditions are (124)-(127), with $\chi_{11} \in [0,1]$, as reported in Lemma 13. These are four equations in four unknowns. The unknowns are $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$ if $\varepsilon^* < \varepsilon^{**}$, or $(\varepsilon^*, \chi_{11}, \varphi, \mathcal{Z})$ if $\varepsilon^* = \varepsilon^{**}$ (recall (15) and Lemma 10 imply $\varepsilon^* \leq \varepsilon^{**}$ in a monetary equilibrium with credit). We consider each case in turn.

(*i*) Suppose $\varepsilon^* < \varepsilon^{**}$. In this case, (125) implies $\varepsilon^{**} = \varepsilon^n$, where $\varepsilon^n \in [\varepsilon_L, \varepsilon_H]$ is the unique solution to $G(\varepsilon^n) = \lambda$. Combined, conditions (126) and (127) imply a single equation in the unknown ε^* that can be written as $T(\varepsilon^*) = 0$, where

$$T(x) \equiv (1-\alpha)\,\theta\,(\varepsilon^n - x) + [\alpha + (1-\alpha)(1-\theta)] \int_x^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \frac{1}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)$$
$$- \iota \Bigg\{ \bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \Bigg\}.$$

Differentiate T and evaluate the derivative at $x = \varepsilon^*$ to obtain

$$T'(\varepsilon^*) = -\left\{ (1-\alpha)\,\theta + [\alpha + (1-\alpha)(1-\theta)] \left\{ [1 - G(\varepsilon^*)] + \iota G(\varepsilon^*) \right\} \right\} < 0.$$

Hence, if there is a ε^* that satisfies $T(\varepsilon^*) = 0$, it is unique. Notice that

$$T(\varepsilon_L) = (1-\alpha)\,\theta\,(\varepsilon^n - \varepsilon_L) + [\alpha + (1-\alpha)(1-\theta)](\bar{\varepsilon} - \varepsilon_L) + (1-\alpha)\,\theta \frac{1}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)$$
$$- \iota \left\{ \bar{\varepsilon} + (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \right\},$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\iota}(\lambda)$, where $\bar{\iota}(\lambda)$ is defined in the statement of the proposition. Also,

$$T(\varepsilon^n) = \left[\alpha + (1-\alpha) \left(1 + \theta \frac{\lambda}{1-\lambda} \right) \right] \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)$$
$$- \iota \left\{ \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\,\theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right\},$$

so $T(\varepsilon^n) < 0$ if and only if $\hat{\iota}(\lambda) < \iota$. Thus, if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, there exists a unique ε^* that satisfies $T(\varepsilon^*) = 0$, and $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$. Given ε^* and ε^{**}, φ is given by (127). Finally, given ε^*, ε^{**}, and φ, $\mathcal{Z}$ is given by (124), which can be written as (37). From this expression, it is clear that $0 < \mathcal{Z} \Leftrightarrow \alpha > 0$ and $\varepsilon_L < \varepsilon^*$ (and the latter condition is implied by $\iota < \bar{\iota}(\lambda)$).

(*ii*) Suppose $\varepsilon^* = \varepsilon^{**}$. In this case, (124)-(127) become

$$1 = [1 - G(\varepsilon^*)] \left(\alpha + \frac{1-\alpha}{1-\lambda} \right) \left(\frac{\mathcal{Z}}{\varphi} + 1 \right) \tag{128}$$

$$\chi_{11} = \frac{\lambda}{1-\lambda} \frac{1 - G(\varepsilon^*)}{G(\varepsilon^*)} \tag{129}$$

$$\iota\varphi = \left[\alpha + (1-\alpha) \left(1 - \theta + \theta \frac{1}{1-\lambda} \right) \right] \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) \, dG(\varepsilon) \tag{130}$$

$$\varphi = \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) \, dG(\varepsilon) + (1-\alpha)\theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) \, dG(\varepsilon). \tag{131}$$

Conditions (130) and (131) imply a single equation in the unknown ε^* that can be written as $\mathcal{T}(\varepsilon^*) = 0$, where

$$\begin{aligned} \mathcal{T}(\varepsilon^*) \equiv & \left\{ \alpha + (1-\alpha) \left[1 + (1-\iota)\, \theta \frac{\lambda}{1-\lambda} \right] \right\} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) \, dG(\varepsilon) \\ & - \iota \left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) \, dG(\varepsilon) \right]. \end{aligned}$$

Differentiate $\mathcal{T}$ and evaluate the derivative at the ε^* that solves $\mathcal{T}(\varepsilon^*) = 0$ to obtain

$$\mathcal{T}'(\varepsilon^*) = -\iota \left\{ \frac{\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)}{\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) dG(\varepsilon)} [1 - G(\varepsilon^*)] + G(\varepsilon^*) \right\} \leq 0,$$

with "=" only if $\iota = 0$. Hence, if there is a ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, it is unique. Notice that $\mathcal{T}(\varepsilon_H) = -\iota\varepsilon_H$, so $\mathcal{T}(\varepsilon_H) < 0$ if and only if $0 < \iota$. Also,

$$\begin{aligned} \mathcal{T}(\varepsilon^n) = & \left\{ \alpha + (1-\alpha) \left[1 + (1-\iota)\, \theta \frac{\lambda}{1-\lambda} \right] \right\} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n) \, dG(\varepsilon) \\ & - \iota \left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon) \, dG(\varepsilon) \right], \end{aligned}$$

so $0 \leq \mathcal{T}(\varepsilon^n)$ if and only if $\iota \leq \hat{\iota}(\lambda)$. Thus, if $0 < \iota \leq \hat{\iota}(\lambda)$, there exists a unique ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, and $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\iota}(\lambda)$). Given ε^*, $\chi_{11} \in [0, 1]$ is given by (129) and φ is given by (131). Finally, given ε^* and φ, (128) implies $\mathcal{Z}$. ■

Lemma 15 *The real asset price in the RME is higher than the real asset price in the RNE, i.e.,*

(i) If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then

$$0 < [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) \, dG(\varepsilon) \leq \varphi - \varphi^n. \tag{132}$$

(ii) If $0 < \iota \leq \hat{\iota}(\lambda)$, *then*

$$0 < [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \leq \varphi - \varphi^n. \tag{133}$$

Moreover, in any RME, $\varphi \leq \varepsilon_H$, *with "="* *only if* $\iota = 0$.

Proof. (i) If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, (132) is immediate from (36). (ii) If $0 < \iota \leq \hat{\iota}(\lambda)$, use (32) and the expression for φ in part (ii) of Proposition 2 to write

$$\begin{aligned}
\varphi - \varphi^n = {} & \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\
& + (1-\alpha) \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
& - (1-\alpha)\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right].
\end{aligned}$$

Define

$$\Upsilon(x) \equiv \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{x}^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon) \tag{134}$$

and notice that for all $x \in [\varepsilon^n, \varepsilon_H]$,

$$\Upsilon'(x) = G(x) - \frac{\lambda}{1-\lambda}[1 - G(x)] \geq 0, \text{ with " = " only if } x = \varepsilon^n. \tag{135}$$

Thus, since $0 < \iota \leq \hat{\iota}(\lambda)$ implies $\varepsilon^n \leq \varepsilon^*$, we have

$$\begin{aligned}
\varphi - \varphi^n \geq {} & \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\
& + (1-\alpha) \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
& - (1-\alpha)\theta \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \right],
\end{aligned}$$

which implies (133).

To show that $\varphi \leq \varepsilon_H$, we again consider two cases. First, suppose $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$. In this case,

$$\begin{aligned}
\varphi - \varepsilon_H = {} & \alpha \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\
& + (1-\alpha) \left[\theta \Upsilon(\varepsilon^n) + (1-\theta) \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\
& < 0.
\end{aligned}$$

Second, suppose $0 < \iota \leq \hat{\iota}(\lambda)$. In this case,

$$\begin{aligned}
\varphi - \varepsilon_H = {} & \alpha \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\
& + (1-\alpha) \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + \theta \frac{\lambda}{1-\lambda} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] \\
& \leq \alpha \left[\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) - (\varepsilon_H - \bar{\varepsilon}) \right] + (1-\alpha) \left[\Upsilon(\varepsilon^*) - (\varepsilon_H - \bar{\varepsilon}) \right] \\
& \leq 0.
\end{aligned}$$

To conclude, notice the last inequality is strict unless $\iota \to 0$, which implies $\varepsilon^* \to \varepsilon_H$, and therefore $\varphi \to \varepsilon_H$. ■

Proof of Proposition 3. Part (i) is an immediate corollary of Lemma 15. For part (ii), we consider two cases in turn.

If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then

$$\frac{d\varepsilon^*}{d\iota} = -\frac{\frac{\partial T(\varepsilon^*)}{\partial \iota}}{T'(\varepsilon^*)} = -\frac{-\varphi}{-\left\{(1-\alpha)\theta + [\alpha + (1-\alpha)(1-\theta)]\left\{[1 - G(\varepsilon^*)] + \iota G(\varepsilon^*)\right\}\right\}} < 0,$$

where $T(\cdot)$ is the equilibrium map defined in part (i) of the proof of Proposition 2. Then, from (36),

$$\frac{d\varphi}{d\iota} = [\alpha + (1-\alpha)(1-\theta)]\, G(\varepsilon^*) \frac{d\varepsilon^*}{d\iota} < 0.$$

If $0 < \iota \leq \hat{\iota}(\lambda)$, then

$$\frac{d\varepsilon^*}{d\iota} = -\frac{\frac{\partial \mathcal{T}(\varepsilon^*)}{\partial \iota}}{\mathcal{T}'(\varepsilon^*)} = -\frac{-\varphi}{-\iota \left\{ \frac{\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)}{\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) dG(\varepsilon)} [1 - G(\varepsilon^*)] + G(\varepsilon^*) \right\}} < 0,$$

where $\mathcal{T}(\cdot)$ is the equilibrium map defined in part (ii) of the proof of Proposition 2. Then, differentiating the expression for φ in part (ii) of the statement of Proposition 2,

$$\frac{d\varphi}{d\iota} = \left[G(\varepsilon^*) - (1-\alpha)\theta \frac{\lambda}{1-\lambda} [1 - G(\varepsilon^*)] \right] \frac{d\varepsilon^*}{d\iota}.$$

Notice that

$$
\begin{aligned}
0 &= \left[G(\varepsilon^n) - \frac{\lambda}{1-\lambda}[1 - G(\varepsilon^n)]\right] \\
&\leq \left[G(\varepsilon^*) - \frac{\lambda}{1-\lambda}[1 - G(\varepsilon^*)]\right] \\
&< G(\varepsilon^*) - (1-\alpha)\,\theta\frac{\lambda}{1-\lambda}[1 - G(\varepsilon^*)],
\end{aligned}
$$

where the first inequality follows because $G(x) - \frac{\lambda}{1-\lambda}[1 - G(x)]$ is increasing in x, and $\varepsilon^n \leq \varepsilon^*$ for all $0 < \iota \leq \hat{\iota}(\lambda)$. Hence, $d\varphi/d\iota < 0$. ■

D.7 Cashless limits

Proof of Proposition 4. Without loss of generality, we compute the relevant limits along a trajectory starting from any economy indexed by the (λ, ι) such that $\iota \in [\hat{\iota}(\lambda), \bar{\iota}(\lambda)]$. As $\lambda \to 1$, the mapping T defined in part (i) of the proof of Proposition 2 converges uniformly to the mapping $T_{\lambda=1}$ defined by

$$
\begin{aligned}
T_{\lambda=1}(x) \equiv{}& (1-\alpha)\,\theta\,(\varepsilon_H - x) + [\alpha + (1-\alpha)(1-\theta)]\int_x^{\varepsilon_H}(\varepsilon - x)\,dG(\varepsilon) \\
&- \iota\left\{\bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)]\int_{\varepsilon_L}^{x}(x-\varepsilon)\,dG(\varepsilon) + (1-\alpha)\theta\,(\varepsilon_H - \bar{\varepsilon})\right\}.
\end{aligned}
$$

(This follows from the fact that $\lim_{\lambda\to 1}\frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon - \varepsilon^n)\,dG(\varepsilon) = \lim_{\lambda\to 1}\frac{1-G(\varepsilon^n)}{G'(\varepsilon^n)} = 0$.) Differentiate $T_{\lambda=1}$ and evaluate the derivative at $x = \varepsilon^*$ to obtain

$$
T'_{\lambda=1}(\varepsilon^*) = -\left\{(1-\alpha)\,\theta + [\alpha + (1-\alpha)(1-\theta)]\left\{[1 - G(\varepsilon^*)] + \iota G(\varepsilon^*)\right\}\right\} < 0.
$$

Hence, if there is a ε^* that satisfies $T(\varepsilon^*) = 0$, it is unique. Notice that

$$
T_{\lambda=1}(\varepsilon_L) = [\bar{\varepsilon} + (1-\alpha)\,\theta\,(\varepsilon_H - \bar{\varepsilon})]\,[\bar{\iota}(1) - \iota],
$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\iota}(1)$. Also,

$$
T_{\lambda=1}(\varepsilon_H) = -\iota\varepsilon_H
$$

so $T(\varepsilon_H) < 0$ if and only if $0 < \iota$. Thus, if $0 \leq \iota \leq \bar{\iota}(1)$, there exists a unique ε^* that satisfies $T_{\lambda=1}(\varepsilon^*) = 0$ (or equivalently, (42)), and $\varepsilon^* \in [\varepsilon_L, \varepsilon_H]$. The limiting expressions (39) and (41) are immediate from (37) and (36). Finally, (40) is the limit of the upper branch of (38). ■

Proof of Proposition 5. Without loss of generality, we compute the relevant limits along a trajectory starting from any economy indexed by the (λ, ι) such that $\iota \in [\hat{\iota}(\lambda), \bar{\iota}(\lambda)]$. From part (i) of the proof of Proposition 2, we know that $\varepsilon^* \to \varepsilon_L$ as $\iota \to \bar{\iota}(\lambda)$, so (36) implies (45), (37) implies (43), and the top branch of (38) implies (44). ■

Proof of Proposition 6. The expression for velocity, (38), can be written as

$$\mathcal{V} = \begin{cases} \frac{\{\alpha[1-G(\varepsilon^*)]+1-\alpha\}[\alpha G(\varepsilon^*)+(1-\alpha)\lambda]}{\alpha G(\varepsilon^*)} & \text{if } \hat{\varsigma}(\alpha) < \iota < \bar{\varsigma}(\alpha) \\ \frac{\left(\alpha+\frac{1-\alpha}{1-\lambda}\right)G(\varepsilon^*)[1-G(\varepsilon^*)]}{\alpha G(\varepsilon^*)+\frac{1-\alpha}{1-\lambda}[G(\varepsilon^*)-\lambda]} & \text{if } 0 < \iota \leq \hat{\varsigma}(\alpha). \end{cases} \tag{136}$$

First, notice that $\hat{\varsigma}(\alpha) \leq \bar{\varsigma}(\alpha)$ for all $\alpha \in [0,1]$, with "=" only if $\lambda = 0$. Hereafter, assume $\lambda > 0$, and fix some $\iota \in (0, \bar{\iota}(0))$.

(i) For $\iota \in (\hat{\varsigma}(0), \bar{\varsigma}(0))$ and α small enough, part (i) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z})$, where

$$\begin{aligned} \varphi = \bar{\varepsilon} + \Bigg\{ & (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \\ & + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \Bigg\} \end{aligned} \tag{137}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*)}{\alpha[1-G(\varepsilon^*)] + 1 - \alpha}\varphi, \tag{138}$$

$\varepsilon^{**} = \varepsilon^n$, and ε^* is the unique $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$ that satisfies $\tilde{T}(\varepsilon^*; \alpha) = 0$, where for any $\varepsilon^* \in [\varepsilon_L, \varepsilon_H]$, $\tilde{T}(\cdot; \alpha)$ is a real-valued function defined by

$$\begin{aligned} \tilde{T}(\varepsilon^*; \alpha) \equiv\ & (1-\alpha)\,\theta\,(\varepsilon^n - \varepsilon^*) + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\ & + (1-\alpha)\,\theta \frac{1}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\ & - \iota \Bigg\{ \bar{\varepsilon} + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\ & + (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] \Bigg\}. \end{aligned}$$

As $\alpha \to 0$, the function $\tilde{T}(\cdot;\alpha)$ converges uniformly to

$$\tilde{T}(\varepsilon^*;0) \equiv \theta(\varepsilon^n - \varepsilon^*) + (1-\theta)\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon - \varepsilon^*)\,dG(\varepsilon) + \theta\frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)$$
$$-\iota\left\{\bar{\varepsilon} + (1-\theta)\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)\right.$$
$$\left.+\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]\right\}.$$

Then (50) is equivalent to $\tilde{T}(\varepsilon^*;0)=0$, while (47), (48), and (49) are obtained from (138), (136), and (137), respectively, by taking the limit as $\alpha \to 0$.

(*ii*) For $\iota \in (0, \hat{\varsigma}(0)]$ and α small enough, part (*ii*) of Proposition 2 implies the monetary equilibrium is a vector $(\varepsilon^*, \chi, \varphi, \mathcal{Z})$ that satisfies $\chi = \frac{\lambda}{1-\lambda}\frac{1-G(\varepsilon^*)}{G(\varepsilon^*)}$,

$$\varphi = \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon) + (1-\alpha)\theta\frac{\lambda}{1-\lambda}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon) \tag{139}$$

$$\mathcal{Z} = \frac{\alpha G(\varepsilon^*) + (1-\alpha)\frac{1}{1-\lambda}[G(\varepsilon^*)-\lambda]}{[1-G(\varepsilon^*)]\left[\alpha + (1-\alpha)\frac{1}{1-\lambda}\right]}\varphi, \tag{140}$$

and $\varepsilon^* = \varepsilon^{**}$, where $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\varsigma}(0)$) is the unique solution to $\tilde{\mathcal{T}}(\varepsilon^*;\alpha) = 0$, where for any $\varepsilon^* \in [\varepsilon_L, \varepsilon_H]$, $\tilde{\mathcal{T}}(\cdot;\alpha)$ is a real-valued function defined by

$$\tilde{\mathcal{T}}(\varepsilon^*;\alpha) \equiv \left\{\alpha + (1-\alpha)\left[1 + (1-\iota)\theta\frac{\lambda}{1-\lambda}\right]\right\}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)$$
$$-\iota\left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)\right].$$

As $\alpha \to 0$, the function $\tilde{\mathcal{T}}(\cdot;\alpha)$ converges uniformly to

$$\tilde{\mathcal{T}}(\varepsilon^*;0) \equiv \left[1 + (1-\iota)\theta\frac{\lambda}{1-\lambda}\right]\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon) - \iota\left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)\right].$$

Then (54) is equivalent to $\tilde{\mathcal{T}}(\varepsilon^*;0) = 0$, while while (51), (52), and (53) are obtained from (140), (136), and (139), respectively, by taking the limit as $\alpha \to 0$. ■

D.8 Capital accumulation

Definition 6 *A sequential nonmonetary equilibrium for the economy with investment is an allocation* $\{X_t, A^s_{t+1}\}_{t=0}^{\infty}$ *and a sequence of prices,* $\{\varepsilon^n_t, \phi^s_t, \bar{\phi}^s_t\}_{t=0}^{\infty}$*, that satisfy:* $\bar{\phi}^s_t = \varepsilon^n_t y_t + \phi^s_t$*,* *the law of motion for the capital stock,*

$$A^s_{t+1} = \eta \left(A^s_t + X_t\right),$$

the market-clearing condition for bonds

$$1 = \left[1 - G\left(\varepsilon^n_t\right)\right] \frac{\bar{\phi}^s_t}{\bar{\phi}^s_t - \lambda \phi^s_t},$$

and the individual optimality conditions

$$X_t = X_t\left(\phi^s_t\right)$$

and

$$\phi^s_t = \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon} y_{t+1} + \phi^s_{t+1} + (1-\alpha)\,\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon^n_{t+1}} \left(\varepsilon^n_{t+1} - \varepsilon\right) y_{t+1} dG(\varepsilon)$$
$$+ \frac{\lambda\phi^s_{t+1}}{\bar{\phi}^s_{t+1} - \lambda\phi^s_{t+1}} \int_{\varepsilon^n_{t+1}}^{\varepsilon_H} \left(\varepsilon - \varepsilon^n_{t+1}\right) y_{t+1} dG(\varepsilon)\Bigg]\Bigg\}.$$

Notice that the structure of the equilibrium conditions in Definition 6 is recursive, i.e., one can solve for $\{\varepsilon^n_t, \phi^s_t\}^\infty_{t=0}$ independently of $\left\{X_t, A^s_{t+1}\right\}^\infty_{t=0}$, and then given $\{\phi^s_t\}^\infty_{t=0}$, one gets $\{X_t\}^\infty_{t=0} = \{X_t\left(\phi^s_t\right)\}^\infty_{t=0}$, and given $\{X_t\}^\infty_{t=0}$, $\left\{A^s_{t+1}\right\}^\infty_{t=0}$ follows from the law of motion for the capital stock. Moreover, notice the equations that characterize $\{\varepsilon^n_t, \phi^s_t\}^\infty_{t=0}$ in this economy with endogenous capital accumulation are identical to the conditions that characterize $\{\varepsilon^n_t, \phi^s_t\}^\infty_{t=0}$ in the baseline economy that assumes $A^s_t = A^s$ for all t.

Definition 7 *A sequential monetary equilibrium for the economy with investment is an allocation* $\left\{X_t, A^s_{t+1}\right\}^\infty_{t=0}$ *and a sequence of prices,* $\{\varepsilon^*_t, \varepsilon^{**}_t, p_t, q_t, \phi^m_t, \phi^s_t\}^\infty_{t=0}$*, that satisfy:* $\varepsilon^{**}_t = (p_t\frac{1}{q_t} - \phi^s_t)\frac{1}{y_t}$*,* $\varepsilon^*_t = \left(p_t\phi^m_t - \phi^s_t\right)\frac{1}{y_t}$*,* $\chi_{11} \equiv \chi\left(1,1\right) \in [0,1]$*, the law of motion for capital,*

$$A^s_{t+1} = \eta \left(A^s_t + X_t\right),$$

the market clearing conditions for equity and bonds,

$$0 = \alpha\left[1 - G\left(\varepsilon^*_t\right)\right] \frac{A^m_t + p_t A^s_t}{p_t} + (1-\alpha)\left[1 - G\left(\varepsilon^{**}_t\right)\right] \frac{A^m_t + p_t A^s_t}{p_t - \lambda q_t \phi^s_t} - A^s_t$$
$$0 = \left[1 - \mathbb{I}_{\{q_t\phi^m_t = 1\}}\left(1 - \chi_{11}\right)\right] G\left(\varepsilon^{**}_t\right) - \frac{\lambda q_t \phi^s_t}{p_t - \lambda q_t \phi^s_t}\left[1 - G\left(\varepsilon^{**}_t\right)\right],$$

and the individual optimality conditions,

$$X_t = X_t\left(\phi_t^s\right),$$

and

$$\begin{aligned}
\phi_t^m = \beta\mathbb{E}_t\Bigg\{ & \phi_{t+1}^m + (1-\alpha)\,\theta\left(\frac{1}{q_{t+1}} - \phi_{t+1}^m\right) \\
& + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_{t+1}^*}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^*\right) y_{t+1} \frac{1}{p_{t+1}} dG(\varepsilon) \\
& + (1-\alpha)\,\theta \frac{1}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^s} \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^{**}\right) y_{t+1} dG(\varepsilon) \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\phi_t^s = \beta\eta\mathbb{E}_t\Bigg\{ & \bar{\varepsilon} y_{t+1} + \phi_{t+1}^s + [\alpha + (1-\alpha)(1-\theta)] \int_{\varepsilon_L}^{\varepsilon_{t+1}^*} \left(\varepsilon_{t+1}^* - \varepsilon\right) y_{t+1} dG(\varepsilon) \\
& + (1-\alpha)\,\theta \left[\int_{\varepsilon_L}^{\varepsilon_{t+1}^{**}} \left(\varepsilon_{t+1}^{**} - \varepsilon\right) y_{t+1} dG(\varepsilon) + \frac{\lambda q_{t+1}\phi_{t+1}^s}{p_{t+1} - \lambda q_{t+1}\phi_{t+1}^s} \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H} \left(\varepsilon - \varepsilon_{t+1}^{**}\right) y_{t+1} dG(\varepsilon) \right] \Bigg\}.
\end{aligned}$$

Notice that the structure of the equilibrium conditions in Definition 7 is recursive, i.e., one can solve for prices and marginal valuations independently of $\left\{X_t, A_{t+1}^s\right\}_{t=0}^{\infty}$, and then given $\{\phi_t^s\}_{t=0}^{\infty}$, one gets $\{X_t\}_{t=0}^{\infty} = \{X_t\left(\phi_t^s\right)\}_{t=0}^{\infty}$, and given $\{X_t\}_{t=0}^{\infty}$, $\left\{A_{t+1}^s\right\}_{t=0}^{\infty}$ follows from the law of motion for the capital stock.

Example 1 *Suppose*

$$f_t(n) = \varpi_t n^{\sigma} \tag{141}$$

for $\sigma \in (0,1)$. *Then the optimal amount of general goods that the investor devotes to the production of capital goods is*

$$g_t\left(\phi_t^s\right) = \left(\sigma\varpi_t\phi_t^s\right)^{\frac{1}{1-\sigma}} \tag{142}$$

and the quantity of new capital created by an individual investor is

$$x_t\left(\phi_t^s\right) = \sigma^{\frac{\sigma}{1-\sigma}} \varpi_t^{\frac{1}{1-\sigma}} \left(\phi_t^s\right)^{\frac{\sigma}{1-\sigma}}. \tag{143}$$

Assume

$$\varpi_t = \left(\sigma y_t\right)^{-\sigma}. \tag{144}$$

(i) Consider the baseline discrete-time formulation. Given $\phi_t^s = \phi^s y_t$, (142) and (143) with (144) imply

$$g_t\left(\phi_t^s\right) = \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} y_t$$
$$x_t\left(\phi_t^s\right) = \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}. \tag{145}$$

(ii) Consider the generalized discrete-time economy with period length Δ. Given the asset price is $\Phi_t^s(\Delta) = \Phi^s(\Delta) y_t \Delta$, (142) and (143) with (144) imply

$$g_t\left(\Phi_t^s(\Delta)\right) = \sigma\left[\Phi^s(\Delta)\Delta\right]^{\frac{1}{1-\sigma}} y_t$$
$$x_t\left(\Phi_t^s(\Delta)\right) = \left[\Phi^s(\Delta)\Delta\right]^{\frac{\sigma}{1-\sigma}}$$

and therefore, since $\lim_{\Delta\to 0}\Phi^s(\Delta)\Delta = \phi^s$,

$$\lim_{\Delta\to 0} g_t\left(\Phi_t^s(\Delta)\right) = \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} y_t \tag{146}$$
$$\lim_{\Delta\to 0} x_t\left(\Phi_t^s(\Delta)\right) = \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}. \tag{147}$$

Thus, in the continuous-time approximation, (146) and (147) are the effort rate devoted to investment, and the investment rate, respectively.

Proof of Proposition 7. Notice the equations that characterize prices and marginal valuations in Definitions 6 and 7 are identical to the conditions that characterize prices and marginal valuations in the baseline economy that assumes $A_t^s = A^s$ for all t. Hence, the conditions that characterize prices and marginal valuations in the recursive equilibrium, and in the recursive equilibrium with $\Delta \to 0$, are also the same in the economy with endogenous capital accumulation as in the economy that assumes $A_t^s = A^s$ for all t. Given the production function (141) with (144), the aggregate investment rate is immediate from (147). ■

D.9 Unsecured credit

In this section we develop the model with unsecured credit outlined in Section A.2.

The bargaining solutions for investors without access to credit are as before. The bargaining solutions for investors with access to credit are summarized in the following two results.

Lemma 16 *Consider the economy with no money. If the investor contacts a credit broker, the post-trade portfolio is*

$$\overline{a}_t^s\left(a_t^s,\varepsilon\right)=\chi\left(\varepsilon_t^n,\varepsilon\right)\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right) \tag{148}$$

$$\overline{a}_t^b\left(a_t^s,\varepsilon\right)=\bar{\phi}_t^s\left[a_t^s-\chi\left(\varepsilon_t^n,\varepsilon\right)\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right)\right] \tag{149}$$

and the intermediation fee for the broker is

$$k_t\left(a_t^s,\varepsilon\right)=\left(1-\theta\right)\left(\varepsilon-\varepsilon_t^n\right)y_t\left[\chi\left(\varepsilon_t^n,\varepsilon\right)\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right)-a_t^s\right]. \tag{150}$$

Proof. In a nonmonetary economy, (2) implies $\left[\overline{a}_t^s\left(a_t^s,\varepsilon\right),\overline{a}_t^b\left(a_t^s,\varepsilon\right),k_t\left(a_t^s,\varepsilon\right)\right]$ is the solution to

$$\max_{(\overline{a}_t^s,k_t)\in\mathbb{R}_+^2,\overline{a}_t^b\in\mathbb{R}}\left[\left(\varepsilon y_t+\phi_t^s\right)\left(\overline{a}_t^s-a_t^s\right)+\overline{a}_t^b-k_t\right]^\theta k_t^{1-\theta}$$

$$\text{s.t. }\bar{\phi}_t^s\overline{a}_t^s+\overline{a}_t^b=\bar{\phi}_t^s a_t^s \tag{151}$$

$$-\bar{B}_t\leq\overline{a}_t^b. \tag{152}$$

Notice that the first-order condition with respect to k_t implies (103), so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{\overline{a}_t^s\in\mathbb{R}_+,\overline{a}_t^b\in\mathbb{R}}\left[\left(\varepsilon y_t+\phi_t^s\right)\left(\overline{a}_t^s-a_t^s\right)+\overline{a}_t^b\right]\text{ s.t. (151), and (152).}$$

Since (151) implies $\overline{a}_t^b=\bar{\phi}_t^s(a_t^s-\overline{a}_t^s)$,

$$\overline{a}_t^s\left(a_t^s,\varepsilon\right)=\arg\max_{\overline{a}_t^s}\left(\varepsilon-\varepsilon_t^n\right)\overline{a}_t^s\text{ s.t. }0\leq\overline{a}_t^s\text{ and }\overline{a}_t^s\leq a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}.$$

The solution is given by (148). Given $\overline{a}_t^s\left(a_t^s,\varepsilon\right)$, $\overline{a}_t^b\left(a_t^s,\varepsilon\right)=\bar{\phi}_t^s\left[a_t^s-\overline{a}_t^s\left(a_t^s,\varepsilon\right)\right]$ as in (149), and $k_t\left(a_t^s,\varepsilon\right)$ is given by (103), or equivalently, (150). ■

Lemma 17 *Consider the economy with money, and let* $\bar{\varepsilon}_t^{**}=\max\left(\varepsilon_t^*,\varepsilon_t^{**}\right)$*, where*

$$\varepsilon_t^{**}\equiv\frac{p_t\frac{1}{q_t}-\phi_t^s}{y_t} \tag{153}$$

and ε_t^* *is as defined in (14). Consider an investor who enters the OTC round of period* t *with portfolio* $\boldsymbol{a}_t$ *and valuation* ε *in an economy with money. If the investor contacts a credit broker,* *the post-trade portfolio is*

$$\overline{a}_t^m(\boldsymbol{a}_t,\varepsilon)=\Big\{\mathbb{I}_{\{1<q_t\phi_t^m\}}\left[\mathbb{I}_{\{\varepsilon<\bar{\varepsilon}_t^{**}\}}+\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\chi\left(\bar{\varepsilon}_t^{**},\varepsilon\right)\right]$$
$$+\mathbb{I}_{\{q_t\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon<\bar{\varepsilon}_t^{**}\}}\chi\left(q_t\phi_t^m,1\right)\Big\}\left(a_t^m+p_ta_t^s+q_t\bar{B}_t\right)$$
$$+\mathbb{I}_{\{q_t\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\tilde{a}_t^m \tag{154}$$
$$\overline{a}_t^s(\boldsymbol{a}_t,\varepsilon)=\Big\{\mathbb{I}_{\{\bar{\varepsilon}_t^{**}<\varepsilon\}}+\left[1-\mathbb{I}_{\{q_t\phi_t^m=1\}}\right]\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\chi\left(\bar{\varepsilon}_t^{**},\varepsilon\right)\Big\}\left[a_t^s+\frac{1}{p_t}\left(a_t^m+q_t\bar{B}_t\right)\right]$$
$$+\mathbb{I}_{\{q_t\phi_t^m=1\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\tilde{a}_t^s \tag{155}$$
$$\overline{a}_t^b(\boldsymbol{a}_t,\varepsilon)=-\frac{1}{q_t}\left\{\left[\overline{a}_t^m(\boldsymbol{a}_t,\varepsilon)-a_t^m\right]+p_t\left[\overline{a}_t^s(\boldsymbol{a}_t,\varepsilon)-a_t^s\right]\right\}, \tag{156}$$

where

$$\left(\tilde{a}_t^m,\tilde{a}_t^s\right)\in\left\{\mathbb{R}_+^2:\tilde{a}_t^m+p_t\tilde{a}_t^s\le a_t^m+p_ta_t^s+q_t\bar{B}_t\right\},$$

and the intermediation fee is

$$k_t(\boldsymbol{a}_t,\varepsilon)=(1-\theta)\Big\{\left(\varepsilon y_t+\phi_t^s\right)\left[\overline{a}_t^s(\boldsymbol{a}_t,\varepsilon)-\hat{a}_t^s(\boldsymbol{a}_t,\varepsilon)\right]$$
$$+\phi_t^m\left[\overline{a}_t^m(\boldsymbol{a}_t,\varepsilon)-\hat{a}_t^m(\boldsymbol{a}_t,\varepsilon)\right]+\overline{a}_t^b(\boldsymbol{a}_t,\varepsilon)\Big\}. \tag{157}$$

Proof. With (99), (2) can be written as

$$\max_{(\overline{a}_t^m,\overline{a}_t^s,k_t)\in\mathbb{R}_+^3,\overline{a}_t^b\in\mathbb{R}}\Big\{\left(\varepsilon y_t+\phi_t^s\right)\left[\overline{a}_t^s-\hat{a}_t^s(\boldsymbol{a}_t,\varepsilon)\right]+\phi_t^m\left[\overline{a}_t^m-\hat{a}_t^m(\boldsymbol{a}_t,\varepsilon)\right]+\overline{a}_t^b-k_t\Big\}^\theta k_t^{1-\theta}$$
$$\text{s.t. } \overline{a}_t^m+p_t\overline{a}_t^s+q_t\overline{a}_t^b=a_t^m+p_ta_t^s \tag{158}$$
$$-\bar{B}_t\le\overline{a}_t^b. \tag{159}$$

Notice that the first-order condition with respect to k_t implies (22) so the bargaining solution can be found by solving the following auxiliary problem

$$\max_{(\overline{a}_t^m,\overline{a}_t^s)\in\mathbb{R}_+^2,\overline{a}_t^b\in\mathbb{R}}\Big\{\left(\varepsilon y_t+\phi_t^s\right)\left[\overline{a}_t^s-\hat{a}_t^s(\boldsymbol{a}_t,\varepsilon)\right]+\phi_t^m\left[\overline{a}_t^m-\hat{a}_t^m(\boldsymbol{a}_t,\varepsilon)\right]+\overline{a}_t^b\Big\}$$
$$\text{s.t. (158), and (159).}$$

Once the solution $\overline{a}_t^m(\boldsymbol{a}_t,\varepsilon)$, $\overline{a}_t^s(\boldsymbol{a}_t,\varepsilon)$, and $\overline{a}_t^b(\boldsymbol{a}_t,\varepsilon)$ to this problem has been found, $k_t(\boldsymbol{a}_t,\varepsilon)$ is given by (22). If we use (158) to substitute for $\overline{a}_t^b$, the auxiliary problem is equivalent to

$$\max_{(\overline{a}_t^m,\overline{a}_t^s)\in\mathbb{R}_+^2}\left[\left(\varepsilon y_t+\phi_t^s-\frac{1}{q_t}p_t\right)\overline{a}_t^s+\left(\phi_t^m-\frac{1}{q_t}\right)\overline{a}_t^m\right] \tag{160}$$

$$\text{s.t. } -q_t\bar{B}_t \leq a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right). \tag{161}$$

The Lagrangian corresponding to the auxiliary problem (160) is

$$\begin{aligned}\mathcal{L} = &\left(\varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t\right)\overline{a}_t^s + \left(\phi_t^m - \frac{1}{q_t}\right)\overline{a}_t^m \\ &+ \xi^b\left[a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right) + q_t\bar{B}_t\right] + \xi^m\overline{a}_t^m + \xi^s\overline{a}_t^s,\end{aligned}$$

where ξ^b, ξ^m, and ξ^s are the multipliers on the constraints (161), $0 \leq \overline{a}_t^m$, and $0 \leq \overline{a}_t^s$, respectively. The first-order conditions are

$$\begin{aligned}\varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t + \xi^s - p_t\xi^b = 0 \\ \phi_t^m - \frac{1}{q_t} + \xi^m - \xi^b = 0.\end{aligned}$$

There are eight possible binding patterns for the multipliers $\left(\xi^b, \xi^m, \xi^s\right)$. Case 1. Assume $0 < \xi^m$, $0 < \xi^s$, $0 < \xi^b$. Then $\overline{a}_t^m = \overline{a}_t^s = 0$ and $a_t^m + p_t a_t^s + q_t\bar{B}_t = 0$. Since $0 \leq \bar{B}_t$, this kind of solution has $\overline{a}_t^b = 0$ and is only possible if $a^s = a^m = \bar{B}_t = 0$. Case 2. Assume $0 < \xi^m$, $0 < \xi^s$, $\xi^b = 0$. Then $\overline{a}_t^m = \overline{a}_t^s = 0$, $q_t\overline{a}_t^b = a_t^m + p_t a_t^s$, $\xi^s = \left[\left(\frac{p_t}{q_t} - \phi_t^s\right)\frac{1}{y_t} - \varepsilon\right]y_t$, and $\xi^m = \frac{1}{q_t} - \phi_t^m$. This kind of solution is only possible if $q_t\phi_t^m < 1$ and $\varepsilon y_t < \frac{1}{q_t}p_t - \phi_t^s$. Case 3. Assume $0 < \xi^m$, $\xi^s = 0$, $0 < \xi^b$. Then $\overline{a}_t^m = 0$, $\overline{a}_t^s = a_t^s + \frac{1}{p_t}\left(q_t\bar{B}_t + a_t^m\right)$, $\overline{a}_t^b = -\bar{B}_t$, $p_t\xi^b = \varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t$, and $p_t\xi^m = \varepsilon y_t + \phi_t^s - p_t\phi_t^m$. This kind of solution is only possible if $\max\left(q_t\phi_t^m, 1\right)\frac{1}{q_t}p_t - \phi_t^s < \varepsilon y_t$. Case 4. Assume $\xi^m = 0$, $0 < \xi^s$, $0 < \xi^b$. Then $\overline{a}_t^m = a_t^m + p_t a_t^s + q_t\bar{B}_t$, $\overline{a}_t^s = 0$, $\overline{a}_t^b = -\bar{B}_t$, $\xi^s = p_t\phi_t^m - \phi_t^s - \varepsilon y_t$, and $p_t\xi^b = \left(q_t\phi_t^m - 1\right)\frac{1}{q_t}p_t$. This kind of solution is only possible if $1 < q_t\phi_t^m$ and $\varepsilon y_t < p_t\phi_t^m - \phi_t^s$. Case 5. Assume $0 < \xi^m$, $\xi^s = 0$, $\xi^b = 0$. Then $\overline{a}_t^m = 0$, $\xi^m = \frac{1}{q_t} - \phi_t^m$, and $\left(\overline{a}_t^s, \overline{a}_t^b\right)$ is any pair that satisfies $\left(\overline{a}_t^s, \overline{a}_t^b\right) \in [0, \infty) \times [-\bar{B}_t, \infty)$ and $q_t\overline{a}_t^b + p_t\overline{a}_t^s = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m < 1$ and $\varepsilon y_t = \frac{1}{q_t}p_t - \phi_t^s$. Case 6. Assume $\xi^m = 0$, $\xi^s = 0$, $0 < \xi^b$. Then $p_t\xi^b = \left(q_t\phi_t^m - 1\right)\frac{1}{q_t}p_t = \varepsilon y_t + \phi_t^s - \frac{1}{q_t}p_t$, $\left(\overline{a}_t^m, \overline{a}_t^s\right)$ is any pair that satisfies $\left(\overline{a}_t^m, \overline{a}_t^s\right) \in [0, \infty) \times [0, \infty)$ and $a_t^m - \overline{a}_t^m + p_t\left(a_t^s - \overline{a}_t^s\right) + q_t\bar{B}_t = 0$, and $\overline{a}_t^b = -\bar{B}_t$. This kind of solution is only possible if $1 < q_t\phi_t^m$ and $\varepsilon y_t = p_t\phi_t^m - \phi_t^s$. Case 7. Assume $\xi^m = 0$, $0 < \xi^s$, $\xi^b = 0$. Then $\overline{a}_t^s = 0$, $\xi^s = \frac{1}{q_t}p_t - \phi_t^s - \varepsilon y_t$, and $\left(\overline{a}_t^m, \overline{a}_t^b\right)$ is any pair that satisfies $\left(\overline{a}_t^m, \overline{a}_t^b\right) \in [0, \infty) \times [-\bar{B}_t, \infty)$ and $\overline{a}_t^m + q_t\overline{a}_t^b = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m = 1$ and $\varepsilon y_t < \frac{1}{q_t}p_t - \phi_t^s$. Case 8. Assume $\xi^m = 0$, $\xi^s = 0$, $\xi^b = 0$. Then $\left(\overline{a}_t^m, \overline{a}_t^s, \overline{a}_t^b\right) \in [0, \infty) \times [0, \infty) \times [-\bar{B}_t, \infty)$ is any triple that satisfies $\overline{a}_t^m + p_t\overline{a}_t^s + q_t\overline{a}_t^b = a_t^m + p_t a_t^s$. This kind of solution is only possible if $q_t\phi_t^m = 1$ and $\varepsilon y_t = \frac{1}{q_t}p_t - \phi_t^s$. By collecting the solutions along with the inequality restrictions implied by the eight cases, we obtain (154)-(157). ■

Next, we derive the market-clearing conditions for equity and bonds in the OTC round, in a nonmonetary economy (Lemma 18), and in a monetary economy (Lemma 19).

Lemma 18 *In a nonmonetary equilibrium, the market-clearing condition for equity, $\hat{A}_t^s + \bar{A}_t^s = A^s$ (or bonds, $\bar{A}_t^b = 0$) in the OTC round is:*

$$1 = [1 - G(\varepsilon_t^n)] \left(1 + \frac{N_I \bar{B}_t}{\bar{\phi}_t^s A^s}\right) \tag{162}$$

where

$$\Lambda_t \equiv N_I \bar{B}_t. \tag{163}$$

Proof. The investors' aggregate post-trade holdings of equity in the OTC round of period t are

$$\bar{A}_t^s = (1-\alpha) N_I \int \bar{a}_t^s(a_t, \varepsilon) dH_t(a_t, \varepsilon) = (1-\alpha) [1 - G(\varepsilon_t^n)] \left(A^s + \frac{N_I \bar{B}_t}{\bar{\phi}_t^s}\right)$$
$$\hat{A}_t^s = \alpha N_I \int \hat{a}_t^s(a_t, \varepsilon) dH_t(a_t, \varepsilon) = \alpha A^s$$

and the investors' aggregate post-trade holdings of bonds in the OTC round of period t are

$$\bar{A}_t^b = (1-\alpha) N_I \int \bar{a}_t^b(a_t, \varepsilon) dH_t(a_t, \varepsilon) = (1-\alpha) \bar{\phi}_t^s \left[A^s - [1 - G(\varepsilon_t^n)] \left(A^s + \frac{N_I \bar{B}_t}{\bar{\phi}_t^s}\right)\right].$$

■

Lemma 19 *In a monetary equilibrium, the market-clearing conditions for equity, $\hat{A}_t^s + \bar{A}_t^s = A^s$, and bonds, $\bar{A}_t^b = 0$, in the OTC round are, respectively:*

$$0 = \alpha [1 - G(\varepsilon_t^*)] \frac{A_t^m + p_t A^s}{p_t} + (1-\alpha) [1 - G(\bar{\varepsilon}_t^{**})] \frac{A_t^m + p_t A^s + q_t N_I \bar{B}_t}{p_t} - A^s$$
$$0 = \left\{G(\bar{\varepsilon}_t^{**}) \left[\mathbb{I}_{\{1 < q_t \phi_t^m\}} + \mathbb{I}_{\{q_t \phi_t^m = 1\}} \chi(q_t \phi_t^m, 1)\right] + 1 - G(\bar{\varepsilon}_t^{**})\right\} \frac{A_t^m + p_t A^s + q_t N_I \bar{B}_t}{p_t}$$
$$- \left(A^s + \frac{A_t^m}{p_t}\right).$$

Proof. The investors' aggregate post-trade holdings of equity in the OTC round of period t are

$$\bar{A}_t^s = (1-\alpha) N_I \int \bar{a}_t^s(\boldsymbol{a}_t, \varepsilon) dH_t(\boldsymbol{a}_t, \varepsilon) = (1-\alpha) [1 - G(\bar{\varepsilon}_t^{**})] \left[A^s + \frac{1}{p_t} \left(A_t^m + q_t N_I \bar{B}_t\right)\right]$$
$$\hat{A}_t^s = \alpha N_I \int \hat{a}_t^s(\boldsymbol{a}_t, \varepsilon) dH_t(\boldsymbol{a}_t, \varepsilon) = \alpha [1 - G(\varepsilon_t^*)] \frac{A_t^m + p_t A^s}{p_t}$$

and the investors' aggregate post-trade holdings of bonds in the OTC round of period t are

$$\bar{A}_t^b = (1-\alpha) N_I \int \bar{a}_t^b(\boldsymbol{a}_t,\varepsilon) dH_t(\boldsymbol{a}_t,\varepsilon) = -\frac{p_t}{q_t}(1-\alpha)\left\{\left\{G\left(\bar{\varepsilon}_t^{**}\right)\left[\mathbb{I}_{\{1<q_t\phi_t^m\}} + \mathbb{I}_{\{q_t\phi_t^m=1\}}\chi\left(q_t\phi_t^m,1\right)\right]\right.\right.$$

$$\left.\left. + 1 - G\left(\bar{\varepsilon}_t^{**}\right)\right\}\frac{A_t^m + p_t A^s + q_t N_I \bar{B}_t}{p_t} - \left(A^s + \frac{A_t^m}{p_t}\right)\right\}.$$

■

The following result states that the credit market would be inactive if the net nominal interest rate on bonds, $i_t^m \equiv \frac{1}{q_t\phi_t^m} - 1$, were negative.

Lemma 20 *Consider a monetary equilibrium. If the bond market is active in period t, then $q_t\phi_t^m \leq 1$.*

Proof. In an equilibrium with $1 < q_t\phi_t^m$, the bond-market clearing condition in Lemma 19 becomes

$$0 = \left[A^s + \frac{1}{p_t}\left(A_t^m + q_t N_I \bar{B}_t\right)\right] - \left(A^s + \frac{A_t^m}{p_t}\right).$$

This condition can only hold if $\bar{B}_t = 0$, i.e., if the bond market is inactive at all dates. The condition $1 < q_t\phi_t^m$ implies bond demand is nil, so the bond market can only clear with no trade. ■

In what follows, we focus on monetary equilibria with an active credit market, i.e., equilibria with $q_t\phi_t^m \leq 1$. Notice this implies $\bar{\varepsilon}_t^{**} = \varepsilon_t^{**}$ for all t in any monetary equilibrium.

Next, we derive an investor's value function in a nonmonetary economy (Lemma 21), and in a monetary economy (Lemma 22).

Lemma 21 *Consider an economy without money. The value function of an investor who enters the OTC round of period t with equity holding a_t^s and valuation ε is*

$$V_t\left(a_t^s,\varepsilon\right) = \left[\varepsilon y_t + \phi_t^s + (1-\alpha)\theta\mathbb{I}_{\{\varepsilon<\varepsilon_t^n\}}\left(\varepsilon_t^n - \varepsilon\right)y_t\right]a_t^s + \tilde{W}_t(\varepsilon), \tag{164}$$

where

$$\tilde{W}_t(\varepsilon) \equiv \bar{W}_t + (1-\alpha)\theta\mathbb{I}_{\{\varepsilon_t^n<\varepsilon\}}\left(\varepsilon - \varepsilon_t^n\right)y_t\frac{\bar{B}_t}{\bar{\phi}_t^s}$$

$$\bar{W}_t \equiv \max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left[-\phi_t^s\tilde{a}_{t+1}^s + \beta\mathbb{E}_t\int V_{t+1}\left[\eta\tilde{a}_{t+1}^s + (1-\eta)A^s,\varepsilon\right]dG(\varepsilon)\right].$$

Proof. With (114), and Lemma 16, (5) reduces to

$$V_t\left(a_t^s,\varepsilon\right)=\bar{W}_t+\left(\varepsilon y_t+\phi_t^s\right)a_t^s$$
$$+\left(1-\alpha\right)\theta\left(\varepsilon-\varepsilon_t^n\right)y_t\left[\mathbb{I}_{\{\varepsilon_t^n<\varepsilon\}}\left(a_t^s+\frac{\bar{B}_t}{\bar{\phi}_t^s}\right)-a_t^s\right],$$

which can be written as (164). ■

Lemma 22 *Consider an economy with money. The value function of an investor who enters the OTC round of period t with portfolio $\boldsymbol{a}_t$ and valuation ε is*

$$V_t\left(\boldsymbol{a}_t,\varepsilon\right)=v_{It}^m\left(\varepsilon\right)a_t^m+v_{It}^s\left(\varepsilon\right)a_t^s+\tilde{W}_t\left(\varepsilon\right), \tag{165}$$

where

$$v_{It}^m\left(\varepsilon\right)\equiv\phi_t^m+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\mathbb{I}_{\{\varepsilon_t^*<\varepsilon\}}\left(\varepsilon-\varepsilon_t^*\right)y_t\frac{1}{p_t}$$
$$+\left(1-\alpha\right)\theta\left(\varepsilon-\bar{\varepsilon}_t^{**}\right)y_t\mathbb{I}_{\{\bar{\varepsilon}_t^{**}<\varepsilon\}}\frac{1}{p_t}$$
$$+\left(1-\alpha\right)\theta\left(\frac{1}{q_t}-\phi_t^m\right)\left\{\mathbb{I}_{\{q_t\phi_t^m<1\}}+\mathbb{I}_{\{1<q_t\phi_t^m\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\left[1-2\chi\left(\bar{\varepsilon}_t^{**},\varepsilon\right)\right]\right\}$$
$$v_{It}^s\left(\varepsilon\right)\equiv\varepsilon y_t+\phi_t^s+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\left(\varepsilon_t^*-\varepsilon\right)y_t\mathbb{I}_{\{\varepsilon<\varepsilon_t^*\}}$$
$$+\left(1-\alpha\right)\theta\left(\bar{\varepsilon}_t^{**}-\varepsilon\right)y_t\mathbb{I}_{\{\varepsilon<\bar{\varepsilon}_t^{**}\}}$$
$$+\left(1-\alpha\right)\theta\left(\frac{1}{q_t}-\phi_t^m\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\left[1-2\chi\left(\bar{\varepsilon}_t^{**},\varepsilon\right)\right]p_t$$
$$\tilde{W}_t\left(\varepsilon\right)\equiv\bar{W}_t+\left(1-\alpha\right)\theta\left\{\left(\varepsilon-\bar{\varepsilon}_t^{**}\right)y_t\mathbb{I}_{\{\bar{\varepsilon}_t^{**}<\varepsilon\}}\frac{1}{p_t}\right.$$
$$\left.+\left(\phi_t^m-\frac{1}{q_t}\right)\mathbb{I}_{\{1<q_t\phi_t^m\}}\left\{1+\mathbb{I}_{\{\varepsilon=\bar{\varepsilon}_t^{**}\}}\left[2\chi\left(\bar{\varepsilon}_t^{**},\varepsilon\right)-1\right]\right\}\right\}q_t\bar{B}_t.$$

Proof. With (99), the value function (5) becomes (110), which after substituting $k_t\left(\boldsymbol{a}_t,\varepsilon\right)$ and $\bar{a}_t^b\left(\boldsymbol{a}_t,\varepsilon\right)$ (using (157) and (156), respectively), becomes

$$V_t\left(\boldsymbol{a}_t,\varepsilon\right)=\bar{W}_t+\left(\varepsilon y_t+\phi_t^s\right)a_t^s+\phi_t^m a_t^m$$
$$+\left[\alpha+\left(1-\alpha\right)\left(1-\theta\right)\right]\left\{\left(\varepsilon y_t+\phi_t^s\right)\left[\hat{a}_t^s(\boldsymbol{a}_t,\varepsilon)-a_t^s\right]+\phi_t^m\left[\hat{a}_t^m\left(\boldsymbol{a}_t,\varepsilon\right)-a_t^m\right]\right\}$$
$$+\left(1-\alpha\right)\theta\left\{\left(\varepsilon y_t+\phi_t^s-\frac{1}{q_t}p_t\right)\left[\bar{a}_t^s\left(\boldsymbol{a}_t,\varepsilon\right)-a_t^s\right]+\left(\phi_t^m-\frac{1}{q_t}\right)\left[\bar{a}_t^m\left(\boldsymbol{a}_t,\varepsilon\right)-a_t^m\right]\right\}.$$

Then replace the post-trade allocations $\hat{a}_t^s(\boldsymbol{a}_t, \varepsilon)$ and $\hat{a}_t^m(\boldsymbol{a}_t, \varepsilon)$ (using Lemma 2), and $\bar{a}_t^s(\boldsymbol{a}_t, \varepsilon)$, and $\bar{a}_t^m(\boldsymbol{a}_t, \varepsilon)$ (using Lemma 17), and rearrange terms to arrive at (165). ■

Next, we derive the Euler equations that characterize the investor's optimal portfolio choices in the second subperiod, in a nonmonetary economy (Lemma 23) and in a nonmonetary economy (Lemma 24).

Lemma 23 *Consider an economy with no money. Let $\tilde{a}_{It+1}^s$ denote equity holding chosen by an investor in the second subperiod of period t. Then $\tilde{a}_{It+1}^s$ is optimal if and only if it satisfies*

$$\phi_t^s \geq \beta\eta\mathbb{E}_t\left[\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + (1-\alpha)\theta\int_{\varepsilon_L}^{\varepsilon_{t+1}^n}\left(\varepsilon_{t+1}^n - \varepsilon\right)y_{t+1}dG(\varepsilon)\right] \text{ with “}=\text{” if } \tilde{a}_{It+1}^s > 0.$$

Proof. With (164), the portfolio problem of an investor in the second subperiod (i.e., the maximization on the right side of (3)) can be written as

$$\max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left\{-\phi_t^s + \beta\eta\mathbb{E}_t\left[\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + (1-\alpha)\theta\int_{\varepsilon_L}^{\varepsilon_{t+1}^n}\left(\varepsilon_{t+1}^n - \varepsilon\right)y_{t+1}dG(\varepsilon)\right]\right\}\tilde{a}_{t+1}^s.$$

■

Lemma 24 *Consider an economy with money. Let $\left(\tilde{a}_{It+1}^m, \tilde{a}_{It+1}^s\right)$ denote the portfolio choice of an investor in the second subperiod of period t. The portfolio $\left(\tilde{a}_{It+1}^m, \tilde{a}_{It+1}^s\right)$ is optimal if and only if it satisfies*

$$\left(\phi_t^m - \beta\mathbb{E}_t\bar{v}_{It+1}^m\right)\tilde{a}_{It+1}^m = 0 \leq \phi_t^m - \beta\mathbb{E}_t\bar{v}_{It+1}^m \tag{166}$$

$$\left(\phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{It+1}^s\right)\tilde{a}_{It+1}^s = 0 \leq \phi_t^s - \beta\eta\mathbb{E}_t\bar{v}_{It+1}^s, \tag{167}$$

where

$$\begin{aligned}\bar{v}_{It+1}^m \equiv\ & \phi_{t+1}^m + \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_{t+1}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^*\right)y_{t+1}dG(\varepsilon)\frac{1}{p_{t+1}}\\ &+ (1-\alpha)\theta\frac{1}{p_{t+1}}\int_{\bar{\varepsilon}_{t+1}^{**}}^{\varepsilon_H}\left(\varepsilon - \bar{\varepsilon}_{t+1}^{**}\right)y_{t+1}dG(\varepsilon)\\ &+ (1-\alpha)\theta\left(\frac{1}{q_{t+1}} - \phi_{t+1}^m\right)\mathbb{I}_{\left\{q_{t+1}\phi_{t+1}^m<1\right\}}\end{aligned}$$

and

$$\begin{aligned}\bar{v}_{It+1}^s \equiv\ & \bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_L}^{\varepsilon_{t+1}^*}\left(\varepsilon_{t+1}^* - \varepsilon\right)y_{t+1}dG(\varepsilon)\\ &+ (1-\alpha)\theta\int_{\varepsilon_L}^{\bar{\varepsilon}_{t+1}^{**}}\left(\bar{\varepsilon}_{t+1}^{**} - \varepsilon\right)y_{t+1}dG(\varepsilon).\end{aligned}$$

Proof. With (165), the portfolio problem of an equity broker in the second subperiod (i.e., the maximization on the right side of (3)) can be written as

$$\max_{\tilde{\boldsymbol{a}}_{t+1}\in\mathbb{R}^2_+}\left[-\phi^m_t\tilde{a}^m_{t+1}-\phi^s_t\tilde{a}^s_{t+1}+\beta\mathbb{E}_t\left(\bar{v}^m_{It+1}\tilde{a}^m_{t+1}+\eta\bar{v}^s_{It+1}\tilde{a}^s_{t+1}\right)\right],$$

where $\bar{v}^k_{It+1}\equiv\int v^k_{It+1}(\varepsilon)\,dG(\varepsilon)$ for $k\in\{m,s\}$. ■

Next, we define sequential nonmonetary equilibrium and monetary equilibrium (with an active credit market).

Definition 8 *A (sequential) nonmonetary equilibrium is a sequence $\{\varepsilon^n_t,\phi^s_t,\bar{\phi}^s_t\}^\infty_{t=0}$, that satisfies*

$$\begin{aligned}
0 &= \left[1-G\left(\varepsilon^n_t\right)\right]\left(A^s+\frac{N_I\bar{B}_t}{\varepsilon^n_t y_t+\phi^s_t}\right)-A^s\\
\phi^s_t &= \beta\eta\mathbb{E}_t\left[\bar{\varepsilon}y_{t+1}+\phi^s_{t+1}+(1-\alpha)\,\theta\int_{\varepsilon_L}^{\varepsilon^n_{t+1}}\left(\varepsilon^n_{t+1}-\varepsilon\right)y_{t+1}dG(\varepsilon)\right]\\
\bar{\phi}^s_t &= \varepsilon^n_t y_t+\phi^s_t.
\end{aligned}$$

The first condition in Definition 8 is the bond-market clearing condition (162), the second is the investor's Euler equation from Lemma 23, and the last is the definition of ε^n_t (8).

Definition 9 *A (sequential) monetary equilibrium is a sequence $\{\varepsilon^*_t,\varepsilon^{**}_t,p_t,q_t,\phi^m_t,\phi^s_t\}^\infty_{t=0}$, that satisfy $\varepsilon^{**}_t=(p_t\frac{1}{q_t}-\phi^s_t)\frac{1}{y_t}$, $\varepsilon^*_t=(p_t\phi^m_t-\phi^s_t)\frac{1}{y_t}$, $\chi^B_L\equiv\chi(1,1)\in[0,1]$, the market clearing conditions,*

$$\begin{aligned}
0 &= \alpha\left[1-G\left(\varepsilon^*_t\right)\right]\frac{A^m_t+p_tA^s}{p_t}+(1-\alpha)\left[1-G\left(\varepsilon^{**}_t\right)\right]\frac{A^m_t+p_tA^s+q_tN_I\bar{B}_t}{p_t}-A^s\\
0 &= \left\{G\left(\varepsilon^{**}_t\right)\left[\mathbb{I}_{\{1<q_t\phi^m_t\}}+\mathbb{I}_{\{q_t\phi^m_t=1\}}\chi^B_L\right]+\left[1-G\left(\varepsilon^{**}_t\right)\right]\right\}\frac{A^m_t+p_tA^s+q_tN_I\bar{B}_t}{p_t}\\
&\quad-\left(A^s+\frac{A^m_t}{p_t}\right),
\end{aligned}$$

the Euler equations,

$$\phi_t^m = \beta\mathbb{E}_t\Bigg\{\phi_{t+1}^m + [\alpha + (1-\alpha)(1-\theta)]\int_{\varepsilon_{t+1}^*}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^*\right)y_{t+1}dG(\varepsilon)\frac{1}{p_{t+1}}$$
$$+ (1-\alpha)\theta\frac{1}{p_{t+1}}\int_{\varepsilon_{t+1}^{**}}^{\varepsilon_H}\left(\varepsilon - \varepsilon_{t+1}^{**}\right)y_{t+1}dG(\varepsilon)$$
$$+ (1-\alpha)\theta\left(\frac{1}{q_{t+1}} - \phi_{t+1}^m\right)\mathbb{I}_{\left\{q_{t+1}\phi_{t+1}^m<1\right\}}\Bigg\}$$
$$\phi_t^s = \beta\eta\mathbb{E}_t\Bigg\{\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s + [\alpha + (1-\alpha)(1-\theta)]\int_{\varepsilon_L}^{\varepsilon_{t+1}^*}\left(\varepsilon_{t+1}^* - \varepsilon\right)y_{t+1}dG(\varepsilon)$$
$$+ (1-\alpha)\theta\int_{\varepsilon_L}^{\varepsilon_{t+1}^{**}}\left(\varepsilon_{t+1}^{**} - \varepsilon\right)y_{t+1}dG(\varepsilon)\Bigg\}.$$

Next, we define RNE and RME (with an active credit market). To this end, hereafter we assume $\bar{B}_t$ is as defined in (72). As before, a RNE is a nonmonetary equilibrium in which real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e., $\phi_t^s = \phi^s y_t$ and $\bar{\phi}_t^s = \bar{\phi}^s y_t$ for some $\phi^s, \bar{\phi}^s \in \mathbb{R}_+$. Hence in a RNE, $\varepsilon_t^n = \left(\bar{\phi}_t^s - \phi_t^s\right)\frac{1}{y_t} = \bar{\phi}^s - \phi^s \equiv \varepsilon^n$. Similarly, a RME is a monetary equilibrium in which: (i) real equity prices (general goods per equity share) are time-invariant linear functions of the aggregate dividend, i.e., $\phi_t^s = \phi^s y_t$, $p_t\phi_t^m \equiv \bar{\phi}_{mt}^s = \bar{\phi}_m^s y_t$, and $p_t/q_t \equiv \bar{\phi}_{bt}^s = \bar{\phi}_b^s y_t$ for some $\phi^s, \bar{\phi}_m^s, \bar{\phi}_b^s \in \mathbb{R}_+$; and (ii) real money balances are a constant proportion of output, i.e., $\phi_t^m A_t^m = ZA^s y_t$ for some $Z \in \mathbb{R}_{++}$. Hence in a RME, $\varepsilon_t^* = \left(p_t\phi_t^m - \phi_t^s\right)\frac{1}{y_t} = \bar{\phi}_m^s - \phi^s \equiv \varepsilon^*$, $\varepsilon_t^{**} = \left(p_t/q_t - \phi_t^s\right)\frac{1}{y_t} = \bar{\phi}_b^s - \phi^s \equiv \varepsilon^{**}$, $p_t = \frac{(\varepsilon^* + \phi^s)A_t^m}{ZA^s}$, $\phi_t^m = \frac{ZA^s y_t}{A_t^m}$, and q_t is given by (27).

Definition 10 *A recursive nonmonetary equilibrium of the economy with borrowing limit (72), is a triple* $(\varepsilon^n, \phi^s, \bar{\phi}^s)$*, that satisfies* $\bar{\phi}^s = \varepsilon^n + \phi^s$*,*

$$0 = [1 - G(\varepsilon^n)](1 + \Lambda) - 1$$
$$\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta}\phi^s = \bar{\varepsilon} + (1-\alpha)\theta\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n - \varepsilon)\,dG(\varepsilon).$$

Definition 11 *A recursive monetary equilibrium of the economy with borrowing limit (72), is* *a vector* $(\varepsilon^{*}, \varepsilon^{**}, \phi^{s}, Z, \chi_{L}^{B})$ *that satisfies* $\chi_{L}^{B} \in [0,1]$, *and*

$$0 = \alpha\left[1 - G\left(\varepsilon^{*}\right)\right]\left(1 + \frac{Z}{\varepsilon^{*} + \phi^{s}}\right) + (1-\alpha)\left[1 - G\left(\varepsilon^{**}\right)\right]\left(1 + \Lambda + \frac{Z}{\varepsilon^{*} + \phi^{s}}\right) - 1$$

$$0 = \left\{G\left(\varepsilon^{**}\right)\mathbb{I}_{\{\varepsilon^{*}=\varepsilon^{**}\}}\chi_{L}^{B} + \left[1 - G\left(\varepsilon^{**}\right)\right]\right\}\left(1 + \Lambda + \frac{Z}{\varepsilon^{*} + \phi^{s}}\right) - \left(1 + \frac{Z}{\varepsilon^{*} + \phi^{s}}\right)$$

$$i^{p} = \left[\alpha + (1-\alpha)(1-\theta)\right]\frac{1}{\varepsilon^{*} + \phi^{s}}\int_{\varepsilon^{*}}^{\varepsilon_{H}}\left(\varepsilon - \varepsilon^{*}\right)dG(\varepsilon) + (1-\alpha)\,\theta\frac{1}{\varepsilon^{*} + \phi^{s}}\int_{\varepsilon^{**}}^{\varepsilon_{H}}\left(\varepsilon - \varepsilon^{**}\right)dG(\varepsilon) + (1-\alpha)\,\theta\frac{\varepsilon^{**} - \varepsilon^{*}}{\varepsilon^{*} + \phi^{s}}\mathbb{I}_{\{\varepsilon^{*}<\varepsilon^{**}\}}$$

$$\frac{1 - \bar{\beta}\eta}{\bar{\beta}\eta}\phi^{s} = \bar{\varepsilon} + \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_{L}}^{\varepsilon^{*}}\left(\varepsilon^{*} - \varepsilon\right)dG\left(\varepsilon\right) + (1-\alpha)\,\theta\int_{\varepsilon_{L}}^{\varepsilon^{**}}\left(\varepsilon^{**} - \varepsilon\right)dG\left(\varepsilon\right).$$

In a nonmonetary equilibrium, $p_t/q_t = \bar{\phi}_t^{s} \equiv \varepsilon_t^{n} y_t + \phi_t^{s}$, and therefore the borrowing limit (72) becomes

$$\bar{B}_t \equiv \Lambda\frac{\left(\varepsilon_t^{n} y_t + \phi_t^{n}\right)A^{s}}{N_I}. \tag{168}$$

In a monetary equilibrium, $p_t/q_t = \varepsilon_t^{**} y_t + \phi_t^{s}$, and therefore the borrowing limit (72) becomes

$$\bar{B}_t \equiv \Lambda\frac{\left(\varepsilon_t^{**} y_t + \phi_t^{s}\right)A^{s}}{N_I}. \tag{169}$$

In the discrete-time economy with period length equal to Δ, (168) generalizes to

$$\bar{B}_t\left(\Delta\right) = \Lambda\frac{\left[\varepsilon_t^{n} y_t\Delta + \Phi_t^{n}\left(\Delta\right)\right]A^{s}}{N_I} \tag{170}$$

and (169) generalizes to

$$\bar{B}_t \equiv \Lambda\frac{\left[\varepsilon_t^{**} y_t\Delta + \Phi_t^{s}\left(\Delta\right)\right]A^{s}}{N_I}. \tag{171}$$

In a RNE, $\varepsilon_t^{n} = \varepsilon^{n}$ and $\Phi_t^{n}\left(\Delta\right) = \Phi^{n}\left(\Delta\right)y_t\Delta$, so (170) specializes to

$$\bar{B}_t\left(\Delta\right) = \Lambda\frac{\left[\varepsilon^{n} + \Phi^{n}\left(\Delta\right)\right]A^{s}}{N_I}y_t\Delta.$$

In a RME, $\varepsilon_t^{**} = \varepsilon^{**}$ and $\Phi_t^{s}\left(\Delta\right) = \Phi^{s}\left(\Delta\right)y_t\Delta$, so (171) specializes to

$$\bar{B}_t\left(\Delta\right) = \Lambda\frac{\left[\varepsilon^{**} + \Phi^{s}\left(\Delta\right)\right]A^{s}}{N_I}y_t\Delta.$$

Next, we report the equilibrium conditions for the continuous-time limiting economy as $\Delta \to 0$.

Lemma 25 *Consider the limiting economy (as $\Delta \to 0$) with borrowing limit (72). A recursive nonmonetary equilibrium is a pair (ε^n, φ) that satisfies*

$$G(\varepsilon^n) = \frac{\Lambda}{1+\Lambda}$$
$$\varphi = \bar{\varepsilon} + (1-\alpha)\,\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon).$$

Proof. The first equilibrium condition is immediate from the first condition in Definition 10. The second condition is obtained by recognizing that, in a discrete-time economy with period length Δ, the second condition in Definition 10 is

$$\frac{r+\delta-g+g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\,\Delta = \bar{\varepsilon} + (1-\alpha)\,\theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon)$$

and letting $\Delta \to 0$. ■

Lemma 26 *Consider the limiting economy (as $\Delta \to 0$) with borrowing limit (72). A recursive monetary equilibrium is a vector $(\varepsilon^*, \varepsilon^{**}, \varphi, \mathcal{Z}, \chi_L^B)$ that satisfies $\chi_L^B \in [0,1]$, and*

$$0 = \alpha\left[1-G(\varepsilon^*)\right]\left(1+\frac{\mathcal{Z}}{\varphi}\right) + (1-\alpha)\left[1-G(\varepsilon^{**})\right]\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right) - 1$$
$$0 = \left\{G(\varepsilon^{**})\,\mathbb{I}_{\{\varepsilon^*=\varepsilon^{**}\}}\chi_L^B + \left[1-G(\varepsilon^{**})\right]\right\}\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right) - \left(1+\frac{\mathcal{Z}}{\varphi}\right)$$
$$\iota\varphi = \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta\left[(\varepsilon^{**}-\varepsilon^*)\,\mathbb{I}_{\{\varepsilon^*<\varepsilon^{**}\}} + \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right]$$
$$\varphi = \bar{\varepsilon} + \left[\alpha + (1-\alpha)(1-\theta)\right]\int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\,\theta \int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon).$$

Proof. In a discrete-time economy with period length Δ, the equilibrium conditions in Definition 11 generalize to

$$0 = \alpha\left[1-G(\varepsilon^*)\right]\left(1+\frac{\mathrm{Z}(\Delta)\,\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta}\right)$$
$$+ (1-\alpha)\left[1-G(\varepsilon^{**})\right]\left[1+\Lambda+\frac{\mathrm{Z}(\Delta)\,\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta}\right] - 1$$

$$0 = \left\{G(\varepsilon^{**})\,\mathbb{I}_{\{\varepsilon^*=\varepsilon^{**}\}}\chi_L^B + \left[1-G(\varepsilon^{**})\right]\right\}\left[1+\Lambda+\frac{\mathrm{Z}(\Delta)\,\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta}\right]$$
$$-\left(1+\frac{\mathrm{Z}(\Delta)\,\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\,\Delta}\right)$$

$$\frac{i^p}{\Delta}\Phi^s(\Delta)\Delta = \frac{\Phi^s(\Delta)\Delta}{\varepsilon^*\Delta+\Phi^s(\Delta)\Delta}\Bigg\{[\alpha+(1-\alpha)(1-\theta)]\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)$$

$$+(1-\alpha)\theta\left[(\varepsilon^{**}-\varepsilon^*)\mathbb{I}_{\{\varepsilon^*<\varepsilon^{**}\}}+\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})\,dG(\varepsilon)\right]\Bigg\}$$

$$\frac{r+\delta-g+g\delta\Delta}{(1+g\Delta)(1-\delta\Delta)}\Phi^s(\Delta)\Delta = \bar{\varepsilon}+[\alpha+(1-\alpha)(1-\theta)]\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)$$

$$+(1-\alpha)\theta\int_{\varepsilon_L}^{\varepsilon^{**}}(\varepsilon^{**}-\varepsilon)\,dG(\varepsilon).$$

Take the limit as $\Delta\to 0$ to obtain the conditions in the statement of the lemma. ■

Proof of Proposition 8. As $\alpha\to 0$, the equilibrium conditions in Lemma 26 become

$$0=[1-G(\varepsilon^{**})]\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right)-1 \tag{172}$$

$$0=\left\{G(\varepsilon^{**})\mathbb{I}_{\{\varepsilon^*=\varepsilon^{**}\}}\chi_L^B+[1-G(\varepsilon^{**})]\right\}\left(1+\Lambda+\frac{\mathcal{Z}}{\varphi}\right)-\left(1+\frac{\mathcal{Z}}{\varphi}\right) \tag{173}$$

$$\iota\varphi=(1-\theta)\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)+\theta\int_{\varepsilon^{**}}^{\varepsilon_H}(\varepsilon-\varepsilon^{**})\,dG(\varepsilon)$$
$$+\theta(\varepsilon^{**}-\varepsilon^*)\mathbb{I}_{\{\varepsilon^*<\varepsilon^{**}\}} \tag{174}$$

$$\varphi=\bar{\varepsilon}+(1-\theta)\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon)\,dG(\varepsilon)+\theta\int_{\varepsilon_L}^{\varepsilon^{**}}(\varepsilon^{**}-\varepsilon)\,dG(\varepsilon) \tag{175}$$

where $\chi_L^B\in[0,1]$. These are four equations in four unknowns. The unknowns are $(\varepsilon^*,\varepsilon^{**},\phi^s,Z)$ if $\varepsilon^*<\varepsilon^{**}$, or $(\varepsilon^*,\chi_L^B,\phi^s,Z)$ if $\varepsilon^*=\varepsilon^{**}$. We consider each case in turn.

(i) Suppose $\varepsilon^*<\varepsilon^{**}$. In this case, (172) and (173) imply $\frac{\mathcal{Z}}{\varphi}=0$ and $\varepsilon^{**}=\varepsilon^n$. Combined, conditions (174) and (175) imply a single equation in the unknown ε^* that can be written as $T(\varepsilon^*)=0$, where

$$T(x)\equiv\theta(\varepsilon^n-x)+(1-\theta)\int_x^{\varepsilon_H}(\varepsilon-x)\,dG(\varepsilon)+\theta\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)$$
$$-\iota\left[\bar{\varepsilon}+(1-\theta)\int_{\varepsilon_L}^{x}(x-\varepsilon)\,dG(\varepsilon)+\theta\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)\right].$$

Differentiate T and evaluate the derivative at $x=\varepsilon^*$ to obtain

$$T'(\varepsilon^*)=-\{\theta+(1-\theta)[1-G(\varepsilon^*)+\iota G(\varepsilon^*)]\}<0.$$

Hence, if there is a ε^* that satisfies $T(\varepsilon^*) = 0$, it is unique. Notice that

$$
\begin{aligned}
T(\varepsilon_L) = {} & \theta(\varepsilon^n - \varepsilon_L) + (1-\theta)(\bar{\varepsilon} - \varepsilon_L) + \theta \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \\
& - \iota \left[\bar{\varepsilon} + \theta \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) \right],
\end{aligned}
$$

so $0 < T(\varepsilon_L)$ if and only if $\iota < \bar{\varsigma}_0$. Also,

$$
T(\varepsilon^n) = \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) \right],
$$

so $T(\varepsilon^n) < 0$ if and only if $\hat{\varsigma}_0 < \iota$. Thus, if $\hat{\varsigma}_0 < \iota < \bar{\varsigma}_0$, there exists a unique ε^* that satisfies $T(\varepsilon^*) = 0$, and $\varepsilon^* \in (\varepsilon_L, \varepsilon^n)$. Given ε^* and ε^{**}, φ is given by (172).

(ii) Suppose $\varepsilon^* = \varepsilon^{**}$. In this case, (172)-(175) become

$$
0 = [1 - G(\varepsilon^*)] \left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - 1 \tag{176}
$$

$$
0 = \left\{ G(\varepsilon^*)\, \chi_L^B + [1 - G(\varepsilon^*)] \right\} \left(1 + \Lambda + \frac{\mathcal{Z}}{\varphi}\right) - \left(1 + \frac{\mathcal{Z}}{\varphi}\right) \tag{177}
$$

$$
\iota\varphi = \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \tag{178}
$$

$$
\varphi = \bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon). \tag{179}
$$

Combined, conditions (178) and (179) imply a single equation in the unknown ε^* that can be written as $\mathcal{T}(\varepsilon^*) = 0$, where

$$
\mathcal{T}(x) \equiv \int_{x}^{\varepsilon_H} (\varepsilon - x)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \int_{\varepsilon_L}^{x} (x - \varepsilon)\, dG(\varepsilon) \right].
$$

Differentiate $\mathcal{T}$ and evaluate the derivative at $x = \varepsilon^*$ to obtain

$$
\mathcal{T}'(\varepsilon^*) = -\left[1 - G(\varepsilon^*) + \iota G(\varepsilon^*)\right] < 0.
$$

Hence, if there is a ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, it is unique. Notice that

$$
\mathcal{T}(\varepsilon^n) = \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) - \iota \left[\bar{\varepsilon} + \int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) \right],
$$

so $0 \leq \mathcal{T}(\varepsilon^n)$ if and only if $\iota \leq \hat{\varsigma}_0$. Also,

$$
\mathcal{T}(\varepsilon_H) = -\iota\varepsilon_H \leq 0, \text{ with “ = ” only if } \iota = 0.
$$

Thus, if $0 < \iota \leq \hat{\varsigma}_0$, there exists a unique ε^* that satisfies $\mathcal{T}(\varepsilon^*) = 0$, and $\varepsilon^* \in [\varepsilon^n, \varepsilon_H)$ (with $\varepsilon^* = \varepsilon^n$ only if $\iota = \hat{\varsigma}_0$). Given ε^*, φ is given by (179). Given ε^* and φ, (176) implies

$$\mathcal{Z} = \frac{G(\varepsilon^*) - [1 - G(\varepsilon^*)]\Lambda}{1 - G(\varepsilon^*)}\varphi.$$

Finally, given, ε^*, φ, and $\mathcal{Z}$, (177) implies

$$\chi_L^B = 1 - \frac{1 - G(\varepsilon^*)}{G(\varepsilon^*)}\Lambda.$$

■

D.10 Efficiency

Proof of Proposition 9. The constraint (76) must bind for every t at an optimum, so the planner's problem is equivalent to

$$\max_{\{\tilde{a}_{t+1}^I, \bar{a}_t^I\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}_t^I(d\varepsilon) N_I$$

$$\text{s.t. (74), (77), and } \int_{\varepsilon_L}^{\varepsilon_H} \bar{a}_t^I(d\varepsilon) \leq a_t^I.$$

Then clearly,

$$W^*(y_0) \leq \varepsilon_H A^s \left(\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t\right). \tag{180}$$

The allocation consisting of $\tilde{a}_t^I = A^s/N_I$ and the Dirac measure $\bar{a}_t^I(E) = \frac{A^s}{N_I}\mathbb{I}_{\{\varepsilon_H \in E\}}$ defined in the statement of the proposition achieve the value on the right side of (180) and therefore solve the planner's problem. Notice that $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t = \frac{\bar{\beta}}{1-\bar{\beta}} y_0$, so

$$W^*(y_0) = \frac{\bar{\beta}}{1-\bar{\beta}}\varepsilon_H A^s y_0.$$

Hence, in the discrete-time economy with period of length Δ, welfare is

$$\mathcal{W}^*(y_0) = \frac{1 + g\Delta}{(r-g)\Delta}\varepsilon_H A^s y_0 \Delta.$$

Rearrange this expression and take the limit as $\Delta \to 0$ to arrive at (78). ■

Proof of Proposition 10. The constraint (94) must bind for every t at an optimum, so the planner's problem is equivalent to

$$\max_{\{\tilde{a}_{t+1}^I, \bar{a}_t^I, h_{2t}^I, X_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\int_{\varepsilon_L}^{\varepsilon_H} \varepsilon y_t \bar{a}_t^I(d\varepsilon) - h_{2t}^I\right] N_I$$

$$\text{s.t. (91), (92), (93), and } \int_{\varepsilon_L}^{\varepsilon_H} \bar{a}_t^I(d\varepsilon) \leq a_t^I.$$

Clearly,

$$W^*(A_0^s, y_0) \leq \max_{\{h_{2t}^I\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^\infty \beta^t \left(\varepsilon_H A_t^s y_t - h_{2t}^I N_I\right) \text{ s.t. } A_{t+1}^s = \eta\left[A_t^s + f_t\left(h_{2t}^I\right) N_I\right]. \tag{181}$$

Once $\left\{h_{2t}^I\right\}_{t=0}^\infty$ has been found, we can use (93) to get $X_t = f_t\left(h_{2t}^I\right) N_I$, and (91) at equality to get $\tilde{a}_{t+1}^I = \frac{A_t^s + X_t}{N_I}$. Let $\bar{W}^*(A_0, y_0)$ denote the value of the right side of (181); it satisfies

$$\bar{W}^*(A_t^s, y_t) = \max_{0 \leq h}\left[\varepsilon_H A_t^s y_t - h N_I + \beta \mathbb{E}_t \bar{W}^*\left(A_{t+1}^s, y_{t+1}\right)\right] \tag{182}$$
$$\text{s.t. } A_{t+1}^s = \eta\left[A_t^s + f_t(h) N_I\right].$$

It is easy to show the optimal value function that satisfies (182) is $\bar{W}^*(A_t^s, y_t) = (B A_t^s + C) y_t$, where

$$B = \frac{\varepsilon_H}{1 - \bar{\beta}\eta}$$
$$C = \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\varepsilon_H\right)^{\frac{1}{1-\sigma}} N_I.$$

The decision rule implied by (182) is

$$h(y_t) = \sigma\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\varepsilon_H\right)^{\frac{1}{1-\sigma}} y_t \tag{183}$$

and the implied aggregate investment is

$$f_t[h(y_t)] N_I = \left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\varepsilon_H\right)^{\frac{\sigma}{1-\sigma}} N_I. \tag{184}$$

Hence,

$$\bar{W}^*(A_t^s, y_t) = \left(\frac{\varepsilon_H}{1-\bar{\beta}\eta} A_t^s + \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\varepsilon_H\right)^{\frac{1}{1-\sigma}} N_I\right) y_t. \tag{185}$$

The OTC-market allocation consisting of the Dirac measure $\bar{a}_t^I(E) = \frac{A_t^s}{N_I}\mathbb{I}_{\{\varepsilon_H \in E\}}$ defined in the statement of the proposition along with the decision rules (183) and (184) achieve the value on the right side of (181) and therefore solve the planner's problem, i.e., $W^*(A_t^s, y_t) = \bar{W}^*(A_t^s, y_t)$.

Next consider the generalization to a time period of length Δ. In this case, (182) becomes

$$\bar{W}^*(A_t^s, y_t) = \max_{0 \leq h}\left[\varepsilon_H A_t^s y_t \Delta - \Delta h N_I + \beta \mathbb{E}_t \bar{W}^*\left(A_{t+\Delta}^s, y_{t+\Delta}\right)\right] \tag{186}$$
$$\text{s.t. } A_{t+\Delta}^s = \eta\left[A_t^s + \Delta f_t(h) N_I\right],$$

where y_t, h, and $f_t(h)$ are now the per-unit-time dividend, effort, and output, respectively. It is easy to verify that the optimal value function is still $\bar{W}^*(A_t^s, y_t) = (BA_t^s + C)\, y_t$ (proportional to the *dividend rate*), but with

$$B = \frac{1}{1-\bar{\beta}\eta}\varepsilon_H \Delta = \frac{1}{1-\frac{(1+g\Delta)(1-\delta\Delta)}{1+r\Delta}}\varepsilon_H \Delta = \frac{1+r\Delta}{r+\delta-g+\delta g\Delta}\varepsilon_H$$

$$C = \frac{1}{1-\bar{\beta}}(1-\sigma)\left(\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\varepsilon_H\Delta\right)^{\frac{1}{1-\sigma}} N_I\Delta = \frac{1+r\Delta}{r-g}(1-\sigma)\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r+\delta-g+g\delta\Delta}\varepsilon_H\right]^{\frac{1}{1-\sigma}} N_I.$$

The decision rule for the effort rate is $h(y_t) = \sigma\left(\bar{\beta}\eta B\right)^{\frac{1}{1-\sigma}} y_t$ and the implied aggregate investment rate is $f_t[h(y_t)]\, N_I = \left(\bar{\beta}\eta B\right)^{\frac{\sigma}{1-\sigma}}$, or explicitly,

$$h(y_t) = \sigma\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r-g+\delta+g\delta\Delta}\varepsilon_H\right]^{\frac{1}{1-\sigma}} y_t$$

$$f_t[h(y_t)]\, N_I = \left[\frac{(1+g\Delta)(1-\delta\Delta)}{r-g+\delta+g\delta\Delta}\varepsilon_H\right]^{\frac{\sigma}{1-\sigma}} N_I.$$

Hence,

$$\bar{W}^*(A_t^s, y_t) = \left\{\frac{1+r\Delta}{r+\delta-g+\delta g\Delta}\varepsilon_H A_t^s + \frac{1+r\Delta}{r-g}(1-\sigma)\left[\frac{(1+g\Delta)(1-\delta\Delta)}{r+\delta-g+g\delta\Delta}\varepsilon_H\right]^{\frac{1}{1-\sigma}} N_I\right\} y_t.$$

Take the limit as $\Delta \to 0$ and let $\mathcal{W}^*(A_t^s, y_t) = \lim_{\Delta\to 0} \bar{W}^*(A_t^s, y_t)$ to arrive at (96). ■

Proof of Proposition 11. From Proposition 2, we know that $\varepsilon^* = \varepsilon^{**} \to \varepsilon_H$, and $\varphi \to \varepsilon_H$ as $\iota \to 0$. ■

D.11 Equilibrium welfare

The following result characterizes equilibrium welfare for the economy with exogenous capital.

Lemma 27 *Consider the limiting economy as $\Delta \to 0$ with exogenous capital. Along the path of the recursive equilibrium, we have:*

(i) If the equilibrium is nonmonetary, the welfare function is

$$\mathcal{V}^n(y_t) = \frac{\varphi_1^n}{r-g} A^s y_t \tag{187}$$

with

$$\varphi_1^n \equiv \bar{\varepsilon} + (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right].$$

(ii) If the equilibrium is monetary, the welfare function is

$$\mathcal{V}^m(\mathcal{Z}, y_t) = \frac{1}{r-g}\left(u_1^z \frac{\mathcal{Z}}{\varphi} + \bar{\varepsilon} + u_1^s\right) A^s y_t, \tag{188}$$

where

$$u_1^z \equiv \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) + (1-\alpha)\left[\varepsilon^{**} - \varepsilon^* + \frac{1}{1-\lambda}\int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right]$$

$$u_1^s \equiv \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda}\int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right].$$

Proof. (i) Consider an economy with no money. From (112), the beginning-of-period expected discounted utility of an investor along a recursive equilibrium where he holds a^s equity shares at the beginning of every period is

$$\begin{aligned}\int V_t(a^s, \varepsilon)\, dG(\varepsilon) = &\left\{\bar{\varepsilon} + (1-\alpha)\theta\left[\frac{\varepsilon^n + \phi^n}{\varepsilon^n + (1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) - (\bar{\varepsilon} - \varepsilon^n)\right]\right\} a^s y_t \\ &+ \beta \mathbb{E}_t \int V_{t+1}(a^s, \varepsilon)\, dG(\varepsilon).\end{aligned}$$

Notice we can write $\int V_t(a^s, \varepsilon)\, dG(\varepsilon) = \bar{V}(a^s)\, y_t$, where $\bar{V}(a^s)$ is given by

$$\begin{aligned}\left(1-\bar{\beta}\right)\bar{V}(a^s) = \Bigg\{&\bar{\varepsilon} + (1-\alpha)\theta\Bigg[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) \\ &+ \frac{\lambda\phi^n}{\varepsilon^n + (1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\Bigg]\Bigg\} a^s.\end{aligned} \tag{189}$$

Since there are N_I investors, along a recursive equilibrium path each investor is holding $a^s = A^s/N_I$, and the sum of expected utility across all investors is $N_I \bar{V}(A^s/N_I)\, y_t = \bar{V}(A^s)\, y_t$.

The expected discounted utility of a broker at the beginning of a period is given by (111), i.e.,

$$V_t^B = \alpha^B \int k_t(a_t^s, \varepsilon)\, dH_t(a_t^s, \varepsilon) + \beta \mathbb{E}_t V_{t+1}^B. \tag{190}$$

Since there are N_B bond brokers, the sum of expected utility across all bond brokers is

$$\begin{aligned}N_B V_t^B &= \alpha^B N_B \int k_t(a_t^s, \varepsilon)\, dH_t(a_t^s, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B \\ &= (1-\alpha) N_I \int k_t(a_t^s, \varepsilon)\, dH_t(a_t^s, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B.\end{aligned}$$

From (12), $N_I \int k_t(a_t^s,\varepsilon)\, dH_t(a_t^s,\varepsilon) = \bar{\Xi}(A^s)\, y_t$, where

$$\bar{\Xi}(A^s) \equiv (1-\theta)\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda\phi^n}{\varepsilon^n + (1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right] A^s. \tag{191}$$

Hence, we can write $N_B V_t^B = \bar{V}^B(A^s)\, y_t$ and therefore (190) implies

$$\left(1-\bar{\beta}\right)\bar{V}^B(A^s) = (1-\alpha)\,\bar{\Xi}(A^s). \tag{192}$$

Along a RNE path, total welfare can be written as $V_t = \sum_{k\in\{B,I\}} \bar{V}^k(A^s)\, y_t$ (equity brokers earn no fees so their utility is zero and they contribute nothing to welfare). Combine (189) and (192) to obtain

$$V_t = \frac{1}{1-\bar{\beta}}\left\{\bar{\varepsilon} + (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda\phi^n}{\varepsilon^n + (1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\right]\right\} A^s y_t.$$

In the discrete-time economy with time-period of length Δ, the expression for V_t generalizes to

$$\begin{aligned} V_t = \frac{1+r\Delta}{(r-g)\,\Delta}\Bigg\{\bar{\varepsilon} + (1-\alpha)\Bigg[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) \\ + \frac{\lambda\Phi^n(\Delta)}{\varepsilon^n + (1-\lambda)\Phi^n(\Delta)}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon)\Bigg]\Bigg\} A^s y_t \Delta. \end{aligned}$$

Take the limit as $\Delta \to 0$ and let $\mathcal{V}^n(y_t) \equiv \lim_{\Delta\to 0} V_t$ to arrive at (187).

(ii) Consider a monetary economy. From (109), the beginning-of-period expected welfare of an investor along a recursive equilibrium where he holds portfolio (a_t^m, a^s) at the beginning of every period is

$$\int V_t^I(a_t^m, a^s, \varepsilon)\, dG(\varepsilon) = \bar{v}_{It}^m a_t^m + \bar{v}_{It}^s a^s + \bar{W}_t^I, \tag{193}$$

where $\bar{W}_t^I$ is given by (100), and $\bar{v}_{It}^m$ and $\bar{v}_{It}^s$ are defined in Lemma 5 and can be written as

$$\begin{aligned} \bar{v}_{It}^m &= \bar{v}^z \frac{1}{p_t} y_t \\ \bar{v}_{It}^s &= \bar{v}^s y_t, \end{aligned}$$

where

$$
\begin{aligned}
\bar{v}^{z} \equiv\ & \varepsilon^{*}+\phi^{s}+(1-\alpha)\,\theta\left(\varepsilon^{**}-\varepsilon^{*}\right) \\
&+[\alpha+(1-\alpha)(1-\theta)] \int_{\varepsilon^{*}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{*}\right) dG(\varepsilon) \\
&+(1-\alpha)\,\theta \frac{\varepsilon^{**}+\phi^{s}}{\varepsilon^{**}+(1-\lambda)\,\phi^{s}} \int_{\varepsilon^{**}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{**}\right) dG(\varepsilon) \qquad (194)\\
\bar{v}^{s} \equiv\ & \bar{\varepsilon}+\phi^{s}+[\alpha+(1-\alpha)(1-\theta)] \int_{\varepsilon_{L}}^{\varepsilon^{*}}\left(\varepsilon^{*}-\varepsilon\right) dG(\varepsilon) \\
&+(1-\alpha)\,\theta\left[\int_{\varepsilon_{L}}^{\varepsilon^{**}}\left(\varepsilon^{**}-\varepsilon\right) dG(\varepsilon)+\frac{\lambda \phi^{s}}{\varepsilon^{**}+(1-\lambda)\,\phi^{s}} \int_{\varepsilon^{**}}^{\varepsilon_{H}}\left(\varepsilon-\varepsilon^{**}\right) dG(\varepsilon)\right]. \qquad (195)
\end{aligned}
$$

Along the path of a recursive equilibrium an individual investor is holding portfolio $(a_{t+1}^{m}, a^{s}) = \left(A_{t+1}^{m}/N_{I}, A^{s}/N_{I}\right)$ at the end of period t (and at the beginning of period $t+1$). Therefore,

$$
\bar{W}_{t}^{I}=T_{t}-\phi_{t}^{m} \frac{A_{t+1}^{m}}{N_{I}}-\phi_{t}^{s} \frac{A^{s}}{N_{I}}+\beta \mathbb{E}_{t} \int V_{t+1}^{I}\left(A_{t+1}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon). \qquad (196)
$$

Substitute (196) into (193), and use the government budget constraint, $N_{I} T_{t}=\phi_{t}^{m}\left(A_{t+1}^{m}-A_{t}^{m}\right)$, to get the sum of expected utility across all investors

$$
\begin{aligned}
N_{I} \int V_{t}^{I}\left(A_{t}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon)= & \ \bar{v}^{z} \frac{1}{p_{t}} A_{t}^{m} y_{t}+\bar{v}^{s} y_{t} A^{s}-\phi_{t}^{m} A_{t}^{m}-\phi_{t}^{s} A^{s} \\
& +\beta \mathbb{E}_{t} N_{I} \int V_{t+1}^{I}\left(A_{t+1}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon).
\end{aligned}
$$

Then, since in a recursive equilibrium, $p_{t}=\frac{\left(\varepsilon^{*}+\phi^{s}\right) A_{t}^{m}}{Z A^{s}}$ and $\phi_{t}^{m} A_{t}^{m}=Z A^{s} y_{t}$, we have

$$
\begin{aligned}
N_{I} \int V_{t}^{I}\left(A_{t}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon)= & \left(\frac{\bar{v}^{z}}{\varepsilon^{*}+\phi^{s}}-1\right) Z A^{s} y_{t}+\left(\bar{v}^{s}-\phi^{s}\right) A^{s} y_{t} \\
& +\beta \mathbb{E}_{t} N_{I} \int V_{t+1}^{I}\left(A_{t+1}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon).
\end{aligned}
$$

Hence, we can write $N_{I} \int V_{t}^{I}\left(A_{t}^{m}/N_{I}, A^{s}/N_{I}, \varepsilon\right) dG(\varepsilon)=\bar{V}^{I}\left(Z, A^{s}\right) y_{t}$, and therefore

$$
\bar{V}^{I}\left(Z, A^{s}\right)=\left(\frac{\bar{v}^{z}-\varepsilon^{*}-\phi^{s}}{\varepsilon^{*}+\phi^{s}} Z+\bar{v}^{s}-\phi^{s}\right) A^{s}+\bar{\beta} \bar{V}^{I}\left(Z, A^{s}\right)
$$

so

$$
\left(1-\bar{\beta}\right) \bar{V}^{I}\left(Z, A^{s}\right)=\left(\frac{u^{z}}{\varepsilon^{*}+\phi^{s}} Z+\bar{\varepsilon}+u^{s}\right) A^{s}, \qquad (197)
$$

where

$$u^z \equiv \bar{v}^z - (\varepsilon^* + \phi^s) \tag{198}$$

$$u^s \equiv \bar{v}^s - (\bar{\varepsilon} + \phi^s), \tag{199}$$

with $\bar{v}^z$ and $\bar{v}^s$ given by (194) and (195).

The expected welfare of a broker at the beginning of a period is given by (108), i.e.,

$$V_t^B = \alpha^B \int k_t(\boldsymbol{a}_t, \varepsilon)\, dH_t(\boldsymbol{a}_t, \varepsilon) + \beta \mathbb{E}_t V_{t+1}^B. \tag{200}$$

Since there are N_B bond brokers, the sum of expected utility across all bond brokers is

$$\begin{aligned} N_B V_t^B &= \alpha^B N_B \int k_t(\boldsymbol{a}_t, \varepsilon)\, dH_t(\boldsymbol{a}_t, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B \\ &= (1-\alpha) N_I \int k_t(\boldsymbol{a}_t, \varepsilon)\, dH_t(\boldsymbol{a}_t, \varepsilon) + \beta \mathbb{E}_t N_B V_{t+1}^B. \end{aligned}$$

From (22),

$$\begin{aligned} N_I \int k_t(\boldsymbol{a}_t, \varepsilon)\, dH_t(\boldsymbol{a}_t, \varepsilon) = (1-\theta) \Bigg[& \frac{p_t}{p_t - \lambda q_t \phi_t^s} \int_{\varepsilon_t^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon_t^{**})\, dG(\varepsilon) \\ & + (\varepsilon_t^{**} - \varepsilon_t^*) - \int_{\varepsilon_t^*}^{\varepsilon_H} (\varepsilon - \varepsilon_t^*)\, dG(\varepsilon) \Bigg] \frac{1}{p_t} (A_t^m + p_t A^s) y_t. \end{aligned}$$

In a recursive equilibrium, $N_I \int k_t(\boldsymbol{a}_t, \varepsilon)\, dH_t(\boldsymbol{a}_t, \varepsilon) = \bar{\Xi}(Z, A^s) y_t$, where

$$\begin{aligned} \bar{\Xi}(Z, A^s) = (1-\theta) \Bigg[& \frac{\varepsilon^{**} + \phi^s}{\varepsilon^{**} + (1-\lambda)\phi^s} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon) \\ & + (\varepsilon^{**} - \varepsilon^*) - \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \Bigg] \left(\frac{Z}{\varepsilon^* + \phi^s} + 1 \right) A^s. \end{aligned} \tag{201}$$

Hence, we can write $N_B V_t^B = \bar{V}^B(Z, A^s) y_t$ and therefore (200) implies

$$\left(1 - \bar{\beta}\right) \bar{V}^B(Z, A^s) = (1-\alpha) \bar{\Xi}(Z, A^s). \tag{202}$$

Notice that (201) can be used to write (198) and (199) as

$$u^z = \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) + (1-\alpha)\theta \frac{\bar{\Xi}(Z, A^s)}{(1-\theta)\left(\frac{Z}{\varepsilon^* + \phi^s} + A^s\right)} \tag{203}$$

$$u^s = \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) + (1-\alpha)\theta \frac{\bar{\Xi}(Z, A^s)}{(1-\theta)\left(\frac{Z}{\varepsilon^* + \phi^s} + A^s\right)}. \tag{204}$$

Along a RNE, total welfare is $V_t = \sum_{k \in \{B,I\}} \bar{V}^k (Z, A^s) \, y_t$. With (197) and (202), we obtain

$$V_t = \frac{1}{1-\bar{\beta}} \left[\left(\frac{u^z}{\varepsilon^* + \phi^s} Z + \bar{\varepsilon} + u^s \right) A^s + (1-\alpha) \, \Xi (Z, A^s) \right] y_t$$

and substituting (201), (203) and (204), we arrive at

$$V_t = \frac{1}{1-\bar{\beta}} \left(\tilde{u}_1^z \frac{Z}{\varepsilon^* + \phi^s} + \bar{\varepsilon} + \tilde{u}_1^s \right) A^s y_t$$

with

$$\begin{aligned}
\tilde{u}_1^z &\equiv \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) \, dG(\varepsilon) + (1-\alpha) \left[\varepsilon^{**} - \varepsilon^* + \frac{\varepsilon^{**} + \phi^s}{\varepsilon^{**} + (1-\lambda) \, \phi^s} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}) \, dG(\varepsilon) \right] \\
\tilde{u}_1^s &\equiv \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) \, dG(\varepsilon) \\
&\quad + (1-\alpha) \left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon) \, dG(\varepsilon) + \frac{\lambda \phi^s}{\varepsilon^{**} + (1-\lambda) \, \phi^s} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}) \, dG(\varepsilon) \right].
\end{aligned}$$

For the discrete-time formulation with time-period of length Δ, the expression for V_t generalizes to

$$V_t = \frac{1 + r\Delta}{(r-g)\,\Delta} \left[\tilde{u}_1^z(\Delta) \frac{\mathrm{Z}(\Delta)}{\varepsilon^* + \bar{\Xi}^s(\Delta)} + \bar{\varepsilon} + \tilde{u}_1^s(\Delta) \right] A^s y_t \Delta$$

with

$$\begin{aligned}
\tilde{u}_1^z(\Delta) &\equiv \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*) \, dG(\varepsilon) \\
&\quad + (1-\alpha) \left[\varepsilon^{**} - \varepsilon^* + \frac{\varepsilon^{**} + \Phi^s(\Delta)}{\varepsilon^{**} + (1-\lambda) \, \Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}) \, dG(\varepsilon) \right] \\
\tilde{u}_1^s(\Delta) &\equiv \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) \, dG(\varepsilon) \\
&\quad + (1-\alpha) \left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon) \, dG(\varepsilon) + \frac{\lambda \Phi^s(\Delta)}{\varepsilon^{**} + (1-\lambda) \, \Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**}) \, dG(\varepsilon) \right].
\end{aligned}$$

Take the limit as $\Delta \to 0$ and let $\mathcal{V}^m(\mathcal{Z}, y_t) \equiv \lim_{\Delta \to 0} V_t$ to arrive at (188). ■

The following result characterizes equilibrium welfare for the economy with capital accumulation with production technology given by (69).

Lemma 28 *Consider the limiting economy (as $\Delta \to 0$) with capital accumulation. Along the path of the recursive equilibrium:*

(i) If the equilibrium is nonmonetary, the welfare function is

$$\mathcal{V}^n\left(A_t^s, y_t\right) = \left[\frac{\varphi_1^n}{\rho}A_t^s + \frac{1}{r-g}\left(\frac{\varphi_1^n}{\varphi^n} - \sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right] y_t \tag{205}$$

with φ_1^n *as defined in part (i) of Lemma 27.*

(ii) If the equilibrium is monetary, the welfare function is

$$\begin{aligned} \mathcal{V}^m\left(\mathcal{Z}, A_t^s, y_t\right) &= \frac{1}{r-g}\left\{\left(\frac{u_1^z}{\rho}\frac{\mathcal{Z}}{\varphi} + \frac{\varphi_1}{\rho}\right)\left[(r-g)\,A_t^s + \left(\frac{\varphi}{\rho}\right)^{\frac{\sigma}{1-\sigma}} N_I\right]\right. \\ &\quad \left. -\sigma\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right\} y_t \end{aligned} \tag{206}$$

with $\varphi_1 \equiv \bar{\varepsilon} + u_1^s$ *and* u_1^z *and* u_1^s *as defined in part (ii) of Lemma 27.*

Proof. (i) Consider an economy with no money. From (112), the sum of expected discounted utility across all investors at the beginning of period t along a recursive equilibrium where each investor holds A_t^s/N_I equity shares, is

$$\begin{aligned} N_I \int V_t^I\left(A_t^s/N_I, \varepsilon\right) dG(\varepsilon) = N_I &\left\{\bar{\varepsilon} + \phi^n + (1-\alpha)\,\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right) dG(\varepsilon)\right.\right. \\ &\left.\left. + \frac{\lambda\phi^n}{\varepsilon^n + (1-\lambda)\,\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right) dG(\varepsilon)\right]\right\}\frac{A_t^s}{N_I} y_t + N_I \bar{W}_t^I, \end{aligned}$$

where

$$\begin{aligned} \bar{W}_t^I \equiv &\max_{h_{2t}\in\mathbb{R}_+}\left[\phi^n y_t f_t\left(h_{2t}\right) - h_{2t}\right] \\ &+ \max_{\tilde{a}_{t+1}^s\in\mathbb{R}_+}\left[-\phi^n y_t \tilde{a}_{t+1}^s + \beta\mathbb{E}_t\int V_{t+1}^I\left(\eta\tilde{a}_{t+1}^s, \varepsilon\right) dG(\varepsilon)\right]. \end{aligned}$$

Along a RNE path with $\phi_t^s = \phi^n y_t$, we have $h_{2t} = g_t\left(\phi_t^n\right) = \sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}} y_t$, $f_t\left(h_{2t}\right) = x_t\left(\phi_t^n\right) = \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}$, $\tilde{a}_{t+1}^s = \left(A_t^s + X_t\right)/N_I$, and $X_t = N_I x_t\left(\phi_t^n\right)$, as described in Section A.1 (where as in Section 3), so

$$\bar{W}_t^I \equiv -\left[\sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}} + \phi^n\frac{A_t^s}{N_I}\right] y_t + \beta\mathbb{E}_t\int V_{t+1}^I\left[\eta\left(A_t^s/N_I + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon).$$

Also, along a recursive equilibrium where each investor holds A_t^s/N_I equity shares at the beginning of each period t, the sum of expected utility across all bond brokers in any given period is

$N_B\alpha^B\bar{\Xi}\left(A_t^s/N_I\right)y_t = N_I\left(1-\alpha\right)\bar{\Xi}\left(A_t^s/N_I\right)y_t$, with $\bar{\Xi}\left(\cdot\right)$ as defined in (191). Hence in a RNE, total welfare (the sum of expected utility across all investors and bond brokers), $V\left(A_t^s, y_t\right)$, satisfies the following recursion

$$\begin{aligned}
V\left(A_t^s, y_t\right) = {} & N_I\left\{\bar{\varepsilon} + \phi^n + \left(1-\alpha\right)\theta\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right)dG\left(\varepsilon\right)\right.\right. \\
& \left.\left. + \frac{\lambda\phi^n}{\varepsilon^n + \left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}\frac{A_t^s}{N_I}y_t \\
& + N_I\left(1-\alpha\right)\bar{\Xi}\left(A_t^s/N_I\right)y_t - \left[\sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}} + \phi^n\frac{A_t^s}{N_I}\right]N_I y_t \\
& + \beta\mathbb{E}_t V\left[\eta\left(A_t^s + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}N_I\right), y_{t+1}\right].
\end{aligned}$$

Substitute the expression for $\bar{\Xi}\left(A_t^s/N_I\right)y_t$ to obtain

$$\begin{aligned}
V\left(A_t^s, y_t\right) = {} & \left\{\bar{\varepsilon} + \left(1-\alpha\right)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right)dG\left(\varepsilon\right)\right.\right. \\
& \left.\left. + \frac{\lambda\phi^n}{\varepsilon^n + \left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}A_t^s y_t \\
& - \sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}}N_I y_t + \beta\mathbb{E}_t V\left[\eta\left(A_t^s + \left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}N_I\right), y_{t+1}\right].
\end{aligned} \tag{207}$$

It is easy to show $V\left(A_t^s, y_t\right) = \left(BA_t^s + C\right)y_t$, where

$$\begin{aligned}
\left(1-\bar{\beta}\eta\right)B &= \bar{\varepsilon} + \left(1-\alpha\right)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right)dG\left(\varepsilon\right) + \frac{\lambda\phi^n}{\varepsilon^n + \left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right)dG\left(\varepsilon\right)\right] \\
\left(1-\bar{\beta}\right)C &= \left\{\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\left\{\bar{\varepsilon} + \left(1-\alpha\right)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n - \varepsilon\right)dG\left(\varepsilon\right)\right.\right.\right. \\
& \left.\left.\left. + \frac{\lambda\phi^n}{\varepsilon^n + \left(1-\lambda\right)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon - \varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}\left(\phi^n\right)^{\frac{\sigma}{1-\sigma}} - \sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}}\right\}N_I.
\end{aligned}$$

Hence,

$$\begin{aligned}\left(1-\bar{\beta}\right)V\left(A_t^s,y_t\right)=\;&\frac{1-\bar{\beta}}{1-\bar{\beta}\eta}\left\{\bar{\varepsilon}+(1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)\right.\right.\\&\left.\left.+\frac{\lambda\phi^n}{\varepsilon^n+(1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}A_t^s y_t\\&+\left\{\frac{\bar{\beta}\eta}{1-\bar{\beta}\eta}\left\{\bar{\varepsilon}+(1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)\right.\right.\right.\\&\left.\left.\left.+\frac{\lambda\phi^n}{\varepsilon^n+(1-\lambda)\phi^n}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}\left(\phi^n\right)^{\frac{\sigma}{1-\sigma}}-\sigma\left(\phi^n\right)^{\frac{1}{1-\sigma}}\right\}y_tN_I.\end{aligned}$$

In the economy where the period length is Δ, the recursion (207) generalizes to

$$\begin{aligned}V\left(A_t^s,y_t\right)=\;&\left\{\bar{\varepsilon}+(1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)\right.\right.\\&\left.\left.+\frac{\lambda\Phi^n\left(\Delta\right)}{\varepsilon^n+(1-\lambda)\Phi^n\left(\Delta\right)}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}A_t^s y_t\Delta\\&-\sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}}N_Iy_t\Delta+\beta\mathbb{E}_tV\left[\eta\left(A_t^s+\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}}N_I\Delta\right),y_{t+\Delta}\right],\end{aligned}$$

where $\sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}}y_t$ is the individual effort rate devoted to investment, and $\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}}$ is the individual investment rate. It is easy to show that the value function for this problem is $V\left(A_t^s,y_t\right)=\left[B\left(\Delta\right)A_t^s+C\left(\Delta\right)\right]y_t$ (proportional to the dividend rate, y_t), with

$$\begin{aligned}B\left(\Delta\right)&=\frac{\Delta}{1-\bar{\beta}\eta}\left\{\bar{\varepsilon}+(1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^n}\left(\varepsilon^n-\varepsilon\right)dG\left(\varepsilon\right)+\frac{\lambda\Phi^n\left(\Delta\right)}{\varepsilon^n+(1-\lambda)\Phi^n\left(\Delta\right)}\int_{\varepsilon^n}^{\varepsilon_H}\left(\varepsilon-\varepsilon^n\right)dG\left(\varepsilon\right)\right]\right\}\\C\left(\Delta\right)&=\frac{\Delta}{1-\bar{\beta}}\left[\bar{\beta}\eta B\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{\sigma}{1-\sigma}}-\sigma\left(\Phi^n\left(\Delta\right)\Delta\right)^{\frac{1}{1-\sigma}}\right]N_I.\end{aligned}$$

Notice that

$$\begin{aligned}\lim_{\Delta\to0}B\left(\Delta\right)&=\frac{\varphi_1^n}{\rho}\\\lim_{\Delta\to0}C\left(\Delta\right)&=\frac{1}{r-g}\left(\frac{\varphi_1^n}{\varphi^n}-\sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}}N_I.\end{aligned}$$

Hence, the limiting expression $\mathcal{V}\left(A_t^s,y_t\right)\equiv\lim_{\Delta\to0}V\left(A_t^s,y_t\right)$ is as in (205).

(ii) Consider a monetary economy. From (109), the sum of expected discounted utility across all investors at the beginning of period t along a recursive equilibrium where each investor holds

A_t^m/N_I dollars and A_t^s/N_I equity shares, is

$$N_I \int V_t^I\left(\frac{A_t^m}{N_I},\frac{A_t^s}{N_I},\varepsilon\right) dG\left(\varepsilon\right) = N_I\left(\bar{v}^z \frac{1}{p_t} y_t \frac{A_t^m}{N_I} + \bar{v}^s y_t \frac{A_t^s}{N_I}\right) + N_I \bar{W}_t^I, \tag{208}$$

where $\bar{v}^z$ and $\bar{v}^s$ are given in (194) and (195), and

$$\begin{aligned}\bar{W}_t^I \equiv\, & T_t + \max_{h_{2t}\in\mathbb{R}_+}\left[\phi_t^s f_t\left(h_{2t}\right) - h_{2t}\right] \\ & + \max_{\tilde{\boldsymbol{a}}_{t+1}\in\mathbb{R}_+^2}\left[-\boldsymbol{\phi}_t \tilde{\boldsymbol{a}}_{t+1} + \beta\mathbb{E}_t \int V_{t+1}^I\left(\boldsymbol{a}_{t+1},\varepsilon\right) dG(\varepsilon)\right].\end{aligned}$$

Along a RME path, we have $\phi_t^m A_t^m = Z A_t^s y_t$, $\phi_t^s = \phi^s y_t$, $h_{2t} = g_t\left(\phi_t^s\right) = \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} y_t$, $f_t\left(h_{2t}\right) = x_t\left(\phi_t^s\right) = \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}$, $\tilde{a}_{t+1}^m = A_{t+1}^m/N_I$, $\tilde{a}_{t+1}^s = \left(A_t^s + X_t\right)/N_I$, and $X_t = N_I x_t\left(\phi_t^s\right)$, as described in Section A.1. Also, the government budget constraint is $N_I T_t = \phi_t^m\left(A_{t+1}^m - A_t^m\right)$. Hence,

$$\begin{aligned}\bar{W}_t^I \equiv & -\left[\sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} + \frac{\left(Z+\phi^s\right)A_t^s}{N_I}\right] y_t \\ & + \beta\mathbb{E}_t \int V_{t+1}^I\left[A_{t+1}^m/N_I, \eta\left(A_t^s/N_I + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon).\end{aligned} \tag{209}$$

Substitute (209) into (208) and use the fact that $p_t = \frac{\left(\varepsilon^* + \phi^s\right)A_t^m}{Z A_t^s}$ to get

$$\begin{aligned}N_I \int V_t^I\left(\frac{A_t^m}{N_I},\frac{A_t^s}{N_I},\varepsilon\right) dG\left(\varepsilon\right) = & \left(\frac{\bar{v}^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}^s - \phi^s\right) A_t^s y_t - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t \\ & + \beta\mathbb{E}_t N_I \int V_{t+1}^I\left[\frac{A_{t+1}^m}{N_I}, \eta\left(\frac{A_t^s}{N_I} + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}}\right), \varepsilon\right] dG(\varepsilon).\end{aligned}$$

Also, along a recursive equilibrium where each investor holds portfolio $\left(A_t^m/N_I, A_t^s/N_I\right)$ at the beginning of each period t, the sum of expected utility across all bond brokers in any given period is $N_B \alpha^B \bar{\Xi}\left(Z A_t^s/N_I, A_t^s/N_I\right) y_t = N_I\left(1-\alpha\right)\bar{\Xi}\left(Z A_t^s/N_I, A_t^s/N_I\right) y_t$, with $\bar{\Xi}\left(\cdot,\cdot\right)$ as defined in (201). Hence in a RME, total welfare (the sum of expected utility across all investors and bond brokers), denoted $V\left(Z A_t^s, A_t^s, y_t\right)$, satisfies the following recursion

$$\begin{aligned}V\left(Z A_t^s, A_t^s, y_t\right) = & \left(\frac{\bar{v}^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}^s - \phi^s\right) A_t^s y_t \\ & + N_I\left(1-\alpha\right)\bar{\Xi}\left(Z A_t^s/N_I, A_t^s/N_I\right) y_t - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t \\ & + \beta\mathbb{E}_t V\left[Z\eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), y_{t+1}\right].\end{aligned}$$

Substitute the expression for $\bar{\Xi}\left(Z A_t^s/N_I, A_t^s/N_I\right)$ to obtain

$$\begin{aligned}V\left(Z A_t^s, A_t^s, y_t\right) = & \left(\frac{\bar{v}_1^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}_1^s - \phi^s\right) A_t^s y_t \\ & - \sigma\left(\phi^s\right)^{\frac{1}{1-\sigma}} N_I y_t + \beta\mathbb{E}_t V\left[Z\eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), \eta\left(A_t^s + \left(\phi^s\right)^{\frac{\sigma}{1-\sigma}} N_I\right), y_{t+1}\right],\end{aligned}$$

where

$$
\begin{aligned}
\bar{v}_1^z &\equiv \varepsilon^* + \phi^s + \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
&\quad + (1-\alpha) \left[\varepsilon^{**} - \varepsilon^* + \frac{\varepsilon^{**} + \phi^s}{\varepsilon^{**} + (1-\lambda)\,\phi^s} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon) \right] \\
\bar{v}_1^s &\equiv \bar{\varepsilon} + \phi^s + \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon) \\
&\quad + (1-\alpha) \left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \phi^s}{\varepsilon^{**} + (1-\lambda)\,\phi^s} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon) \right].
\end{aligned}
$$

It is easy to show $V(ZA_t^s, A_t^s, y_t) = (BA_t^s + C)\, y_t$, where

$$
\begin{aligned}
\left(1 - \bar{\beta}\eta\right) B &= \frac{\bar{v}_1^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}_1^s - \phi^s \\
\left(1 - \bar{\beta}\right) C &= \frac{\bar{\beta}\eta}{1 - \bar{\beta}\eta} \left[\frac{\bar{v}_1^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}_1^s - \phi^s \right] (\phi^s)^{\frac{\sigma}{1-\sigma}} N_I - \sigma (\phi^s)^{\frac{1}{1-\sigma}} N_I.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left(1 - \bar{\beta}\right) V(ZA_t^s, A_t^s, y_t) &= \frac{1 - \bar{\beta}}{1 - \bar{\beta}\eta} \left[\frac{\bar{v}_1^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}_1^s - \phi^s \right] A_t^s y_t \\
&\quad + \frac{\bar{\beta}\eta}{1 - \bar{\beta}\eta} \left[\frac{\bar{v}_1^z - \varepsilon^* - \phi^s}{\varepsilon^* + \phi^s} Z + \bar{v}_1^s - \phi^s \right] (\phi^s)^{\frac{\sigma}{1-\sigma}} N_I y_t \\
&\quad - \sigma (\phi^s)^{\frac{1}{1-\sigma}} N_I y_t.
\end{aligned}
$$

In the discrete-time economy where the period length is Δ, this value function generalizes to

$$
\begin{aligned}
&\frac{(r-g)\,\Delta}{1 + r\Delta} V\left(\mathrm{Z}(\Delta)\, \Delta A_t^s, A_t^s, y_t\right) \\
&= \left\{ \left[\frac{\bar{v}_1^z(\Delta) - \varepsilon^* - \Phi^s(\Delta)}{\varepsilon^* + \Phi^s(\Delta)} \mathrm{Z}(\Delta) + \bar{v}_1^s(\Delta) - \Phi^s(\Delta) \right] y_t \Delta \right\} \left\{ \frac{\frac{(r-g)\Delta}{1+r\Delta}}{\frac{r+\delta-g+g\delta\Delta}{1+r\Delta}\Delta} A_t^s \right. \\
&\left. + \frac{(1+g\Delta)(1-\delta\Delta)}{(r+\delta-g+g\delta\Delta)\,\Delta} \left[\Phi^s(\Delta)\,\Delta\right]^{\frac{\sigma}{1-\sigma}} \Delta N_I \right\} \\
&- \sigma \left[\Phi^s(\Delta)\,\Delta\right]^{\frac{1}{1-\sigma}} N_I y_t \Delta,
\end{aligned}
$$

with

$$\bar{v}_1^z(\Delta) \equiv \varepsilon^* + \Phi^s(\Delta) + \alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)$$
$$+ (1-\alpha)\left[\varepsilon^{**} - \varepsilon^* + \frac{\varepsilon^{**} + \Phi^s(\Delta)}{\varepsilon^{**} + (1-\lambda)\Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right]$$
$$\bar{v}_1^s(\Delta) \equiv \bar{\varepsilon} + \Phi^s(\Delta) + \alpha \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon)\, dG(\varepsilon)$$
$$+ (1-\alpha)\left[\int_{\varepsilon_L}^{\varepsilon^{**}} (\varepsilon^{**} - \varepsilon)\, dG(\varepsilon) + \frac{\lambda \Phi^s(\Delta)}{\varepsilon^{**} + (1-\lambda)\Phi^s(\Delta)} \int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right].$$

As usual, $\sigma(\Phi^s(\Delta)\Delta)^{\frac{1}{1-\sigma}} y_t$ is the individual effort *rate* devoted to investment (so the effort accumulated over a period of length Δ is $\sigma(\Phi^s(\Delta)\Delta)^{\frac{1}{1-\sigma}} y_t\Delta$), and $(\Phi^s(\Delta)\Delta)^{\frac{\sigma}{1-\sigma}}$ is the individual investment *rate* (so $(\Phi^s(\Delta)\Delta)^{\frac{\sigma}{1-\sigma}}\Delta$ is the investment accumulated over a period of length Δ). Notice that $\lim_{\Delta\to 0}[\bar{v}_1^z(\Delta) - \varepsilon^* - \Phi^s(\Delta)] = u_1^z$, $\lim_{\Delta\to 0}[\bar{v}_1^s(\Delta) - \Phi^s(\Delta)] = \varphi_1$ (with u_1^z and φ_1 as defined in part (*ii*) of Lemma 27), and $\lim_{\Delta\to 0} \frac{\mathrm{Z}(\Delta)}{\varepsilon^* + \Phi^s(\Delta)} = \frac{\mathcal{Z}}{\varphi}$, so taking the limit as $\Delta \to 0$ and letting $\mathcal{V}(\mathcal{Z}, A_t^s, y_t) \equiv \lim_{\Delta\to 0} V(\mathrm{Z}(\Delta)\Delta A_t^s, A_t^s, y_t)$, we arrive at (206). ■

Proof of Corollary 1. The fact that $\mathcal{V}^n(y_t) \leq \mathcal{V}^m(\mathcal{Z}, y_t)$, with "=" only if $\iota = \bar{\iota}(\lambda)$ is immediate from part (*i*) of Proposition 3 and the fact that $0 \leq \mathcal{Z}$. To show $\mathcal{V}^m(\mathcal{Z}, y_t) \leq \mathcal{W}^*(y_t)$, use (84) to rewrite $\mathcal{V}^m(\mathcal{Z}, y_t)$ as follows

$$\mathcal{V}^m(\mathcal{Z}, y_t) = \frac{1}{r-g}\left[\varepsilon^* + \left(1 + \frac{\mathcal{Z}}{\varphi}\right) u_1^z\right] A^s y_t.$$

Then substitute (82) to get

$$\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z}, y_t) = \varepsilon^* + \left(1 + \frac{\mathcal{Z}}{\varphi}\right)\left\{\alpha \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)\right.$$
$$\left. + (1-\alpha)\left[\varepsilon^{**} - \varepsilon^* + \frac{1}{1-\lambda}\int_{\varepsilon^{**}}^{\varepsilon_H} (\varepsilon - \varepsilon^{**})\, dG(\varepsilon)\right]\right\}.$$

Next we consider two cases. Case 1: If $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by (37), and $\varepsilon^{**} = \varepsilon^n$, and therefore

$$\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z},y_t)=\bar{\varepsilon}+\left\{\frac{\alpha}{[1-G(\varepsilon^*)]\alpha+1-\alpha}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\bar{\varepsilon})\,dG(\varepsilon)\right.$$
$$\left.+\frac{1-\alpha}{[1-G(\varepsilon^*)]\alpha+1-\alpha}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]\right\}$$
$$\leq\bar{\varepsilon}+\left\{\frac{[1-G(\varepsilon^*)]\alpha}{[1-G(\varepsilon^*)]\alpha+1-\alpha}(\varepsilon_H-\bar{\varepsilon})\right.$$
$$\left.+\frac{1-\alpha}{[1-G(\varepsilon^*)]\alpha+1-\alpha}\left[\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)\right]\right\}$$
$$<\varepsilon_H=\frac{r-g}{A^s y_t}\mathcal{W}^*(y_t).$$

The last inequality follows from (134) and (135) that imply

$$\int_{\varepsilon_L}^{\varepsilon^n}(\varepsilon^n-\varepsilon)\,dG(\varepsilon)+\frac{\lambda}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H}(\varepsilon-\varepsilon^n)\,dG(\varepsilon)<\varepsilon_H-\bar{\varepsilon}. \tag{210}$$

Case 2: If $0<\iota\leq\hat{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by the expression in part (ii) of Proposition 2, and $\varepsilon^*=\varepsilon^{**}$, and therefore

$$\frac{r-g}{A^s y_t}\mathcal{V}^m(\mathcal{Z},y_t)=\varepsilon^*+\frac{1}{1-G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon-\varepsilon^*)\,dG(\varepsilon)$$
$$\leq\varepsilon^*+\frac{1}{1-G(\varepsilon^*)}\int_{\varepsilon^*}^{\varepsilon_H}(\varepsilon_H-\varepsilon^*)\,dG(\varepsilon)$$
$$=\varepsilon_H=\frac{r-g}{A^s y_t}\mathcal{W}^*(y_t),$$

where the inequality is strict unless $\iota=0$ (which implies $\varepsilon^*=\varepsilon_H$). ■

Proof of Corollary 3. First, note that

$$(r-g)\left[\mathcal{V}^m(\mathcal{Z},A_t^s,y_t)-\mathcal{V}^n(A_t^s,y_t)\right]\frac{1}{y_t}=\frac{r-g}{\rho}\left(u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1-\varphi_1^n\right)A_t^s$$
$$+\left(\frac{u_1^z\frac{\mathcal{Z}}{\varphi}+\varphi_1}{\varphi}-\sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}}N_I$$
$$-\left(\frac{\varphi_1^n}{\varphi^n}-\sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}}N_I. \tag{211}$$

The first term is strictly positive unless $\iota=\bar{\iota}(\lambda)$ (because $\varphi_1^n\leq\varphi_1$ by Proposition 3, and $0\leq\mathcal{Z}$, and both inequalities are strict unless $\iota=\bar{\iota}(\lambda)$). Hence, to show $\mathcal{V}^n(A_t^s,y_t)\leq\mathcal{V}^m(\mathcal{Z},A_t^s,y_t)$, it is sufficient to show that the sum of the last two terms in (211) is nonnegative (and positive unless $\iota = \bar{\iota}(\lambda)$ and $\theta = 1$). Define

$$\Omega(x, y) \equiv \left(\frac{y}{x} - \sigma\right)\left(\frac{x}{\rho}\right)^{\frac{1}{1-\sigma}}.$$

Notice

$$\frac{\partial}{\partial y}\Omega(x, y) = \frac{1}{x}\left(\frac{x}{\rho}\right)^{\frac{1}{1-\sigma}} > 0 \tag{212}$$

$$\frac{\partial}{\partial x}\Omega(x, y) = \left(\frac{y}{x} - 1\right)\frac{1}{\rho}\frac{\sigma}{1-\sigma}\left(\frac{y}{x}\right)^{\frac{\sigma}{1-\sigma}} > 0 \text{ if and only if } x < y. \tag{213}$$

Then

$$\begin{aligned}
\left(\frac{u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} - \left(\frac{\varphi_1^n}{\varphi^n} - \sigma\right)\left(\frac{\varphi^n}{\rho}\right)^{\frac{1}{1-\sigma}} &\geq \Omega(\varphi, \varphi_1) - \Omega(\varphi^n, \varphi_1^n) \\
&\geq \Omega(\varphi, \varphi_1) - \Omega(\varphi^n, \varphi_1) \\
&\geq \Omega(\varphi^n, \varphi_1) - \Omega(\varphi^n, \varphi_1) = 0.
\end{aligned}$$

The second inequality follows from (212) and the fact that $\varphi_1^n \leq \varphi_1$. The third inequality follows from (213) and the fact that $\varphi^n \leq \varphi \leq \varphi_1$. Thus, $\mathcal{V}^n(A_t^s, y_t) \leq \mathcal{V}^m(\mathcal{Z}, A_t^s, y_t)$, with equality only if $\iota = \bar{\iota}(\lambda)$ and $\theta = 1$ (since in this case, $\mathcal{Z} = 0$ and $\varphi = \varphi_1 = \varphi^n = \varphi_1^n$).

To show that $\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$, proceed as follows. From (96) and (98),

$$\begin{aligned}
&(r-g)\left[\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) - \mathcal{W}^*(A_t^s, y_t)\right]\frac{1}{y_t} \\
&= \frac{r-g}{\rho}\left(u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 - \varepsilon_H\right) A_t^s \\
&+ \left[\left(\frac{u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma\right)\left(\frac{\varphi}{\rho}\right)^{\frac{1}{1-\sigma}} N_I - (1-\sigma)\left(\frac{\varepsilon_H}{\rho}\right)^{\frac{1}{1-\sigma}} N_I\right].
\end{aligned} \tag{214}$$

We first show the first term in (214) is nonpositive (strictly negative unless $\iota = 0$). To this end, we consider two cases in turn. First, if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by (37), and

$\varepsilon^{**} = \varepsilon^n$, and therefore

$$\begin{aligned}
u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 = \bar{\varepsilon} + \Bigg\{ & \frac{\alpha}{[1 - G(\varepsilon^*)]\,\alpha + 1 - \alpha} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \bar{\varepsilon})\, dG(\varepsilon) \\
& + \frac{1-\alpha}{[1 - G(\varepsilon^*)]\,\alpha + 1 - \alpha} \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \Bigg\} \\
\leq \bar{\varepsilon} + \Bigg\{ & \frac{[1 - G(\varepsilon^*)]\,\alpha}{[1 - G(\varepsilon^*)]\,\alpha + 1 - \alpha} (\varepsilon_H - \bar{\varepsilon}) \\
& + \frac{1-\alpha}{[1 - G(\varepsilon^*)]\,\alpha + 1 - \alpha} \left[\int_{\varepsilon_L}^{\varepsilon^n} (\varepsilon^n - \varepsilon)\, dG(\varepsilon) + \frac{\lambda}{1-\lambda} \int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon) \right] \Bigg\} \\
< \varepsilon_H, &
\end{aligned}$$

where the last inequality follows from (134) and (135) that imply (210). Hence, the first term in (214) is negative if $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$. Second, if $0 < \iota \leq \hat{\iota}(\lambda)$, then $\mathcal{Z}/\varphi$ is given by the expression in part (*ii*) of Proposition 2, and $\varepsilon^* = \varepsilon^{**}$, and therefore

$$\begin{aligned}
u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 &= \varepsilon^* + \frac{1}{1 - G(\varepsilon^*)} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon) \\
&\leq \varepsilon^* + \frac{1}{1 - G(\varepsilon^*)} \int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon_H - \varepsilon^*)\, dG(\varepsilon) = \varepsilon_H,
\end{aligned}$$

where the inequality is strict unless $\iota = 0$ (which implies $\varepsilon^* = \varepsilon_H$). Hence, regardless of whether $\hat{\iota}(\lambda) < \iota < \bar{\iota}(\lambda)$ or $0 < \iota \leq \hat{\iota}(\lambda)$, we have $u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 \leq \varepsilon_H$ (with "=" only if $\iota = 0$), so to show $\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$ it is sufficient to show the second term in (214) is nonpositive. This can be shown as follows

$$\begin{aligned}
\left(\frac{u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1}{\varphi} - \sigma \right) \left(\frac{\varphi}{\rho} \right)^{\frac{1}{1-\sigma}} - (1-\sigma) \left(\frac{\varepsilon_H}{\rho} \right)^{\frac{1}{1-\sigma}} &= \Omega\left(\varphi, u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 \right) - \Omega(\varepsilon_H, \varepsilon_H) \\
&\leq \Omega(\varphi, \varepsilon_H) - \Omega(\varepsilon_H, \varepsilon_H) \\
&\leq \Omega(\varepsilon_H, \varepsilon_H) - \Omega(\varepsilon_H, \varepsilon_H) = 0.
\end{aligned}$$

The first inequality follows from (212) and $u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 \leq \varepsilon_H$. The second inequality follows from (213) and $\varphi \leq \varepsilon_H$. Thus, $\mathcal{V}^m(\mathcal{Z}, A_t^s, y_t) \leq \mathcal{W}^*(A_t^s, y_t)$, with "=" only if $\iota = 0$ (since in this case $u_1^z \frac{\mathcal{Z}}{\varphi} + \varphi_1 = \varphi = \varepsilon_H$). ■

D.12 Effects of monetary policy

Proof of Proposition 12. (i) The condition that characterizes ε^* in part (i) of Proposition 2 can be written as

$$\varphi\iota = (1-\alpha)\,\theta\,(\varepsilon^n - \varepsilon^*) + [\alpha + (1-\alpha)\,(1-\theta)]\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon)$$
$$+ (1-\alpha)\,\theta \frac{1}{1-\lambda}\int_{\varepsilon^n}^{\varepsilon_H} (\varepsilon - \varepsilon^n)\, dG(\varepsilon).$$

Totally differentiate this condition with respect to ι to get

$$\varphi + \iota \frac{d\varphi}{d\iota} = -\left\{(1-\alpha)\,\theta + [\alpha + (1-\alpha)\,(1-\theta)]\,[1 - G\,(\varepsilon^*)]\right\} \frac{d\varepsilon^*}{d\iota}. \tag{215}$$

Totally differentiate (36) with respect to ι to get

$$\frac{d\varphi}{d\iota} = [\alpha + (1-\alpha)\,(1-\theta)]\,G\,(\varepsilon^*)\,\frac{d\varepsilon^*}{d\iota}. \tag{216}$$

Together, (215) and (216) imply

$$-\frac{d\varphi}{d\iota}\frac{\iota}{\varphi} = \frac{\iota}{\iota + \frac{(1-\alpha)\theta + [\alpha + (1-\alpha)(1-\theta)][1-G(\varepsilon^*)]}{[\alpha+(1-\alpha)(1-\theta)]G(\varepsilon^*)}}.$$

(ii) The condition that characterizes ε^* in part (ii) of Proposition 2 can be written as

$$\varphi\iota = \left[\alpha + (1-\alpha)\left(1 + \theta\frac{\lambda}{1-\lambda}\right)\right]\int_{\varepsilon^*}^{\varepsilon_H} (\varepsilon - \varepsilon^*)\, dG(\varepsilon).$$

Totally differentiate this condition to get

$$\varphi + \iota \frac{d\varphi}{d\iota} = -\left[\alpha + (1-\alpha)\left(1 + \theta\frac{\lambda}{1-\lambda}\right)\right][1 - G\,(\varepsilon^*)]\,\frac{d\varepsilon^*}{d\iota}. \tag{217}$$

Totally differentiate the expression for φ in part (ii) of Proposition 2 to get

$$\frac{d\varphi}{d\iota} = \left\{G\,(\varepsilon^*) - (1-\alpha)\,\theta\frac{\lambda}{1-\lambda}\,[1 - G\,(\varepsilon^*)]\right\}\frac{d\varepsilon^*}{d\iota}. \tag{218}$$

Combine (217) and (218) to get

$$\frac{d\varphi}{d\iota}\frac{\iota}{\varphi} = -\frac{\iota}{\iota + \frac{\left[\alpha + (1-\alpha)\left(1+\theta\frac{\lambda}{1-\lambda}\right)\right][1-G(\varepsilon^*)]}{G(\varepsilon^*) - (1-\alpha)\theta\frac{\lambda}{1-\lambda}[1-G(\varepsilon^*)]}}.$$

This concludes the proof. ■

E Quantitative robustness

In this section we assess the robustness of the quantitative results of Section 5 to alternative calibration strategies. In our baseline, the parameters α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following three facts: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate, as in the high-frequency empirical estimates in Lagos and Zhang (2019b); (b) transaction velocity of money is 25 per day, which is the average daily number of times a dollar turns over in CHIPS (Clearing House Interbank Payments System); and (c) the median spread on margin loans is about 2.3%, which is the current spread (over the fed funds rate) that a typical prime broker charges a large investor. This procedure delivers $\alpha = .0406$, $\theta = .1612$, and $\Sigma_\varepsilon = 2.0784$. Below, we report results for three alternative calibrations that consider alternative target values for the spread on margin loans and/or velocity.

In the first alternative calibration, denoted (AC1), α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following targets: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is 25 per day; and (c) the median spread on margin loans is about 1.20%. This procedure delivers $\alpha = .0389$, $\theta = .2979$, and $\Sigma_\varepsilon = 2.3653$. Figure 10 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0,1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\lambda = .75$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 11 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0,1] \times \{.10, .30, .70, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\theta = .2979$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 12 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0,1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0389$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 13, 14, and 15 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 13 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as

$\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 14 and 15 tell a similar story. Figure 14, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

In the second alternative calibration, denoted (AC2), α, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with the following targets: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is about 6 per day; and (c) the median spread on margin loans is about 25 basis points. This procedure delivers $\alpha = .0966$, $\theta = .8337$, and $\Sigma_\varepsilon = 2.6429$. Figure 16 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0,1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0966$ and $\lambda = .75$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 17 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0,1] \times \{.10, .25, .83, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0966$ and $\theta = .8337$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 18 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0,1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = .0966$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 19, 20, and 21 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 19 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as $\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 20 and 21 tell a similar story. Figure 20, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

In the third alternative calibration, denoted (AC3), we set $\alpha = 0$, and λ, θ, and Σ_ε are calibrated so that, given the rest of the parametrization, the model is consistent with: (a) the real asset price falls by about 11 basis points in response to a 1 basis point increase in the nominal policy rate; (b) transaction velocity of money is about 25 per day; and (c) the median spread on margin loans is about 25 basis points. This procedure delivers $\lambda = .9159$, $\theta = .8080$, and $\Sigma_\varepsilon = 3.0886$. Figure 22 reports $\mathcal{S}$ for economies indexed by $(\alpha, \lambda) \in [0, 1] \times \{.50, .75, .90, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\lambda = .9159$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and λ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 23 reports $\mathcal{S}$ for economies indexed by $(\alpha, \theta) \in [0, 1] \times \{.10, .25, .80, .99\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\theta = .8080$. As in the baseline calibration, the response of the asset price to nominal rate shocks is sizable for a wide range of values of α and θ, and it is significant even in the pure-credit limiting economy that obtains as $\alpha \to 0$. Figure 24 reports $\mathcal{S}$ for economies indexed by $(\alpha, \rho^p) \in [0, 1] \times \{.03, .04, .0447, .05\}$. The calibration ensures that $\mathcal{S} = 11$ for $\alpha = 0$ and $\rho^p = .0447$. This exercise shows that for every level of α, the asset price response is significant, and tends to be larger in environments with a lower background nominal policy rate. Figures 25, 26, and 27 offer a comprehensive summary of the magnitude of the effects of monetary policy in limiting economies with $\alpha \to 0$. For a wide range of economies indexed by a pair ρ^p and λ, Figure 25 reports the value of $\mathcal{S}$ in the pure-credit limit that obtains as $\alpha \to 0$. The level sets in the right panel show it is not easy to find reasonable parametrizations that imply a value of $\mathcal{S}$ below 5. Figures 26 and 27 tell a similar story. Figure 26, for example, shows that, as predicted by the theory, $\mathcal{S} = 0$ in the pure-credit cashless limit of economies with no credit-market frictions or markups, i.e., economies with $\lambda = \theta = 1$. In contrast, $\mathcal{S}$ is positive and sizable in the pure-credit cashless limit of economies with $\theta < 1$, even if $1 - \theta$ is relatively small.

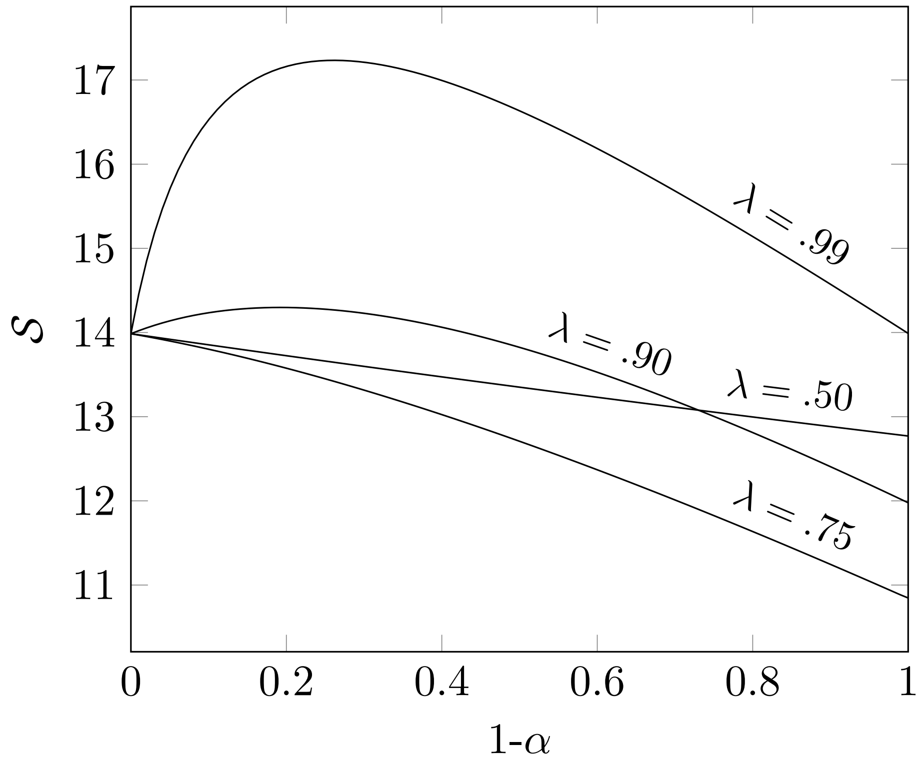


Figure 10: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1 - \alpha$.

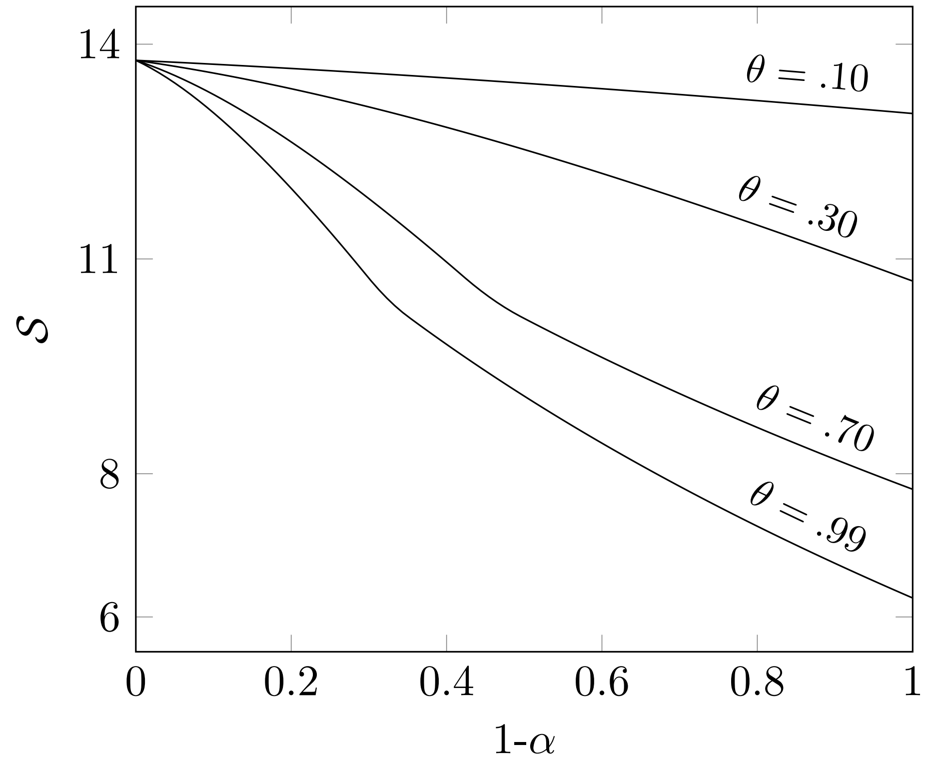


Figure 11: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1-\theta$, and access to credit, $1-\alpha$.

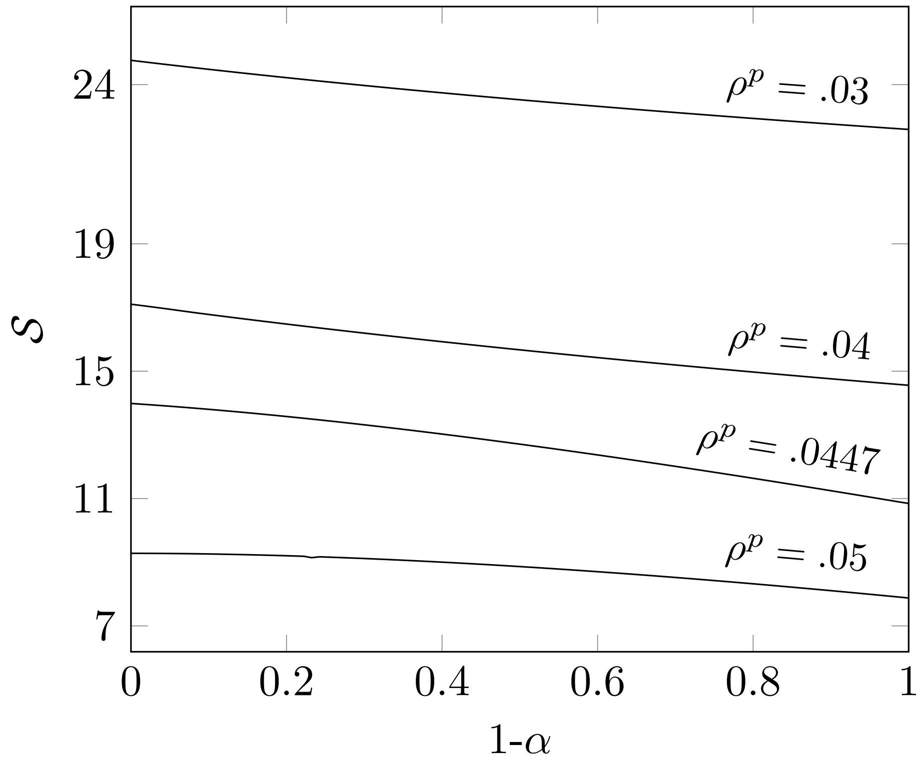


Figure 12: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1 - \alpha$.

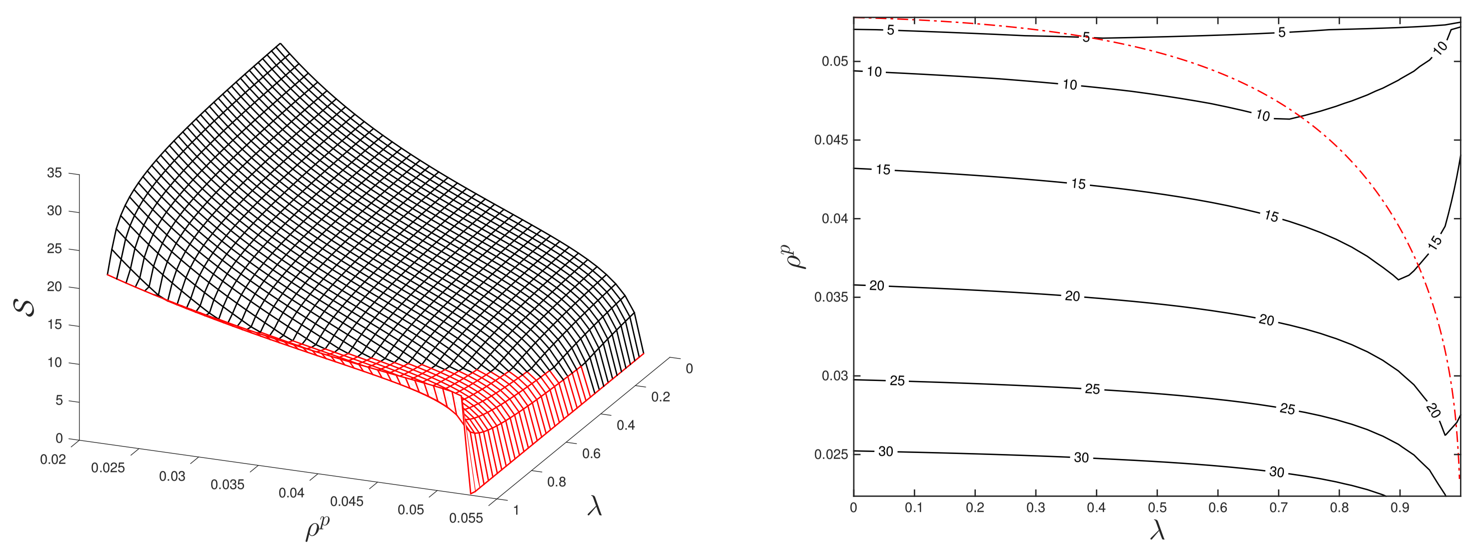


Figure 13: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

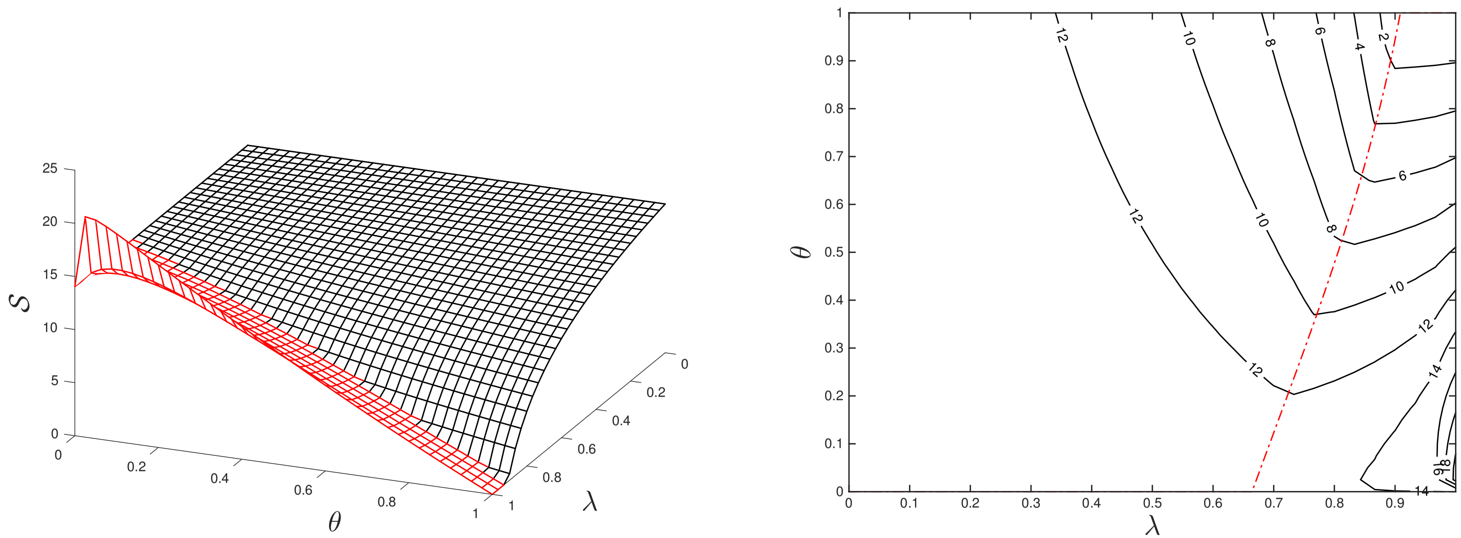


Figure 14: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).

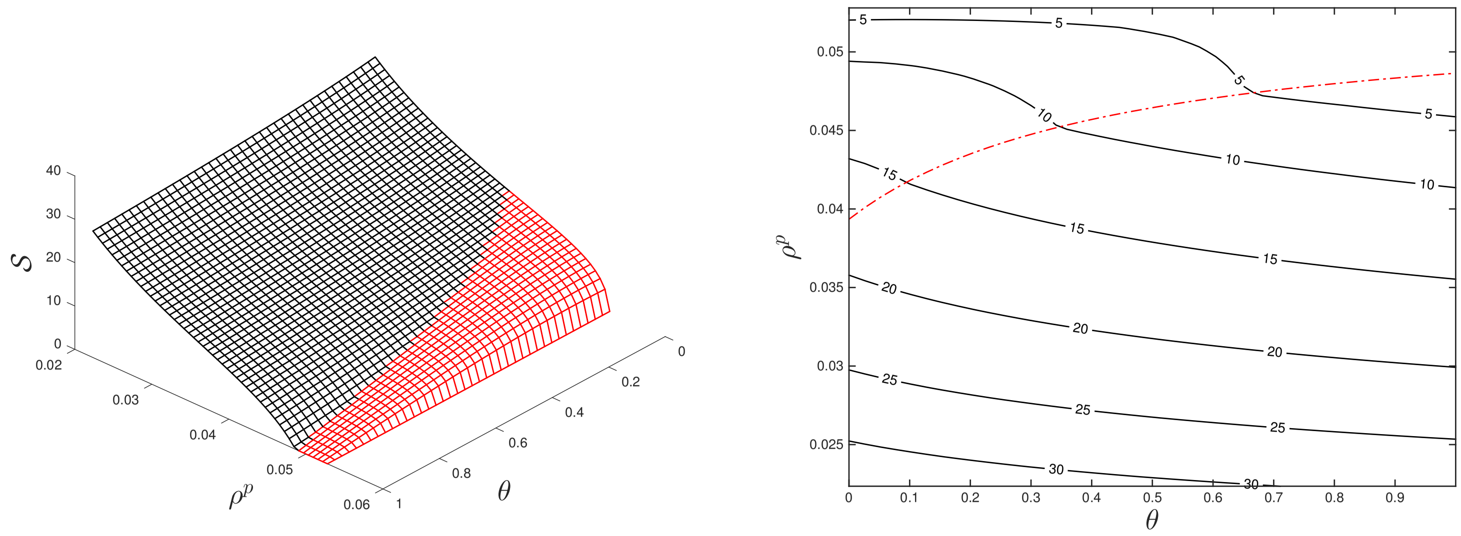


Figure 15: Calibration (AC1): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

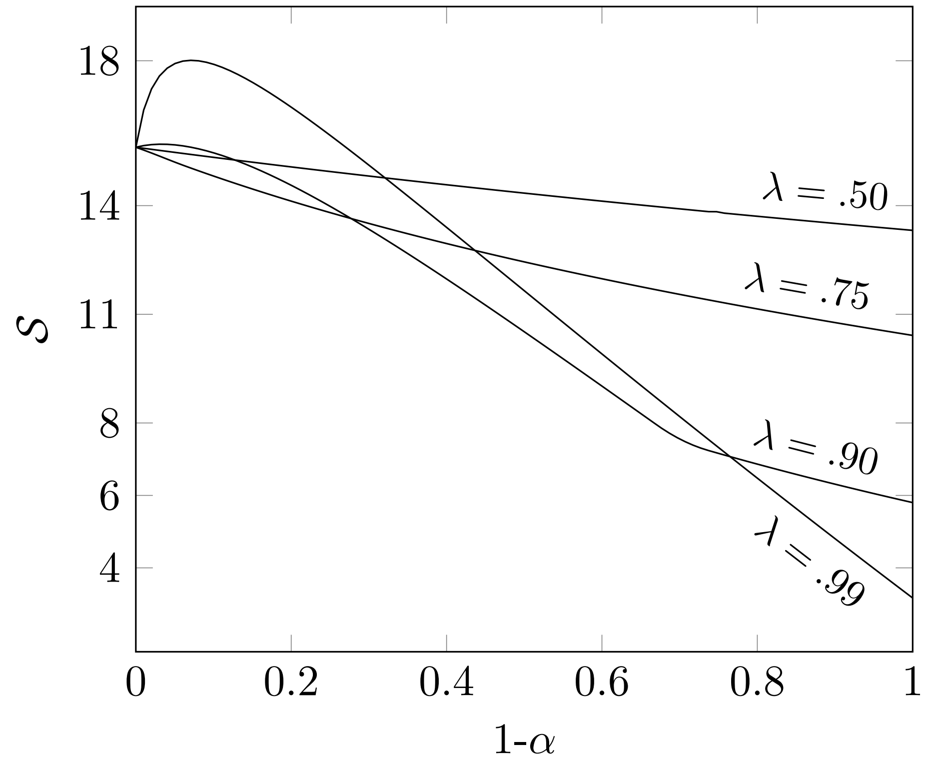


Figure 16: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1 - \alpha$.

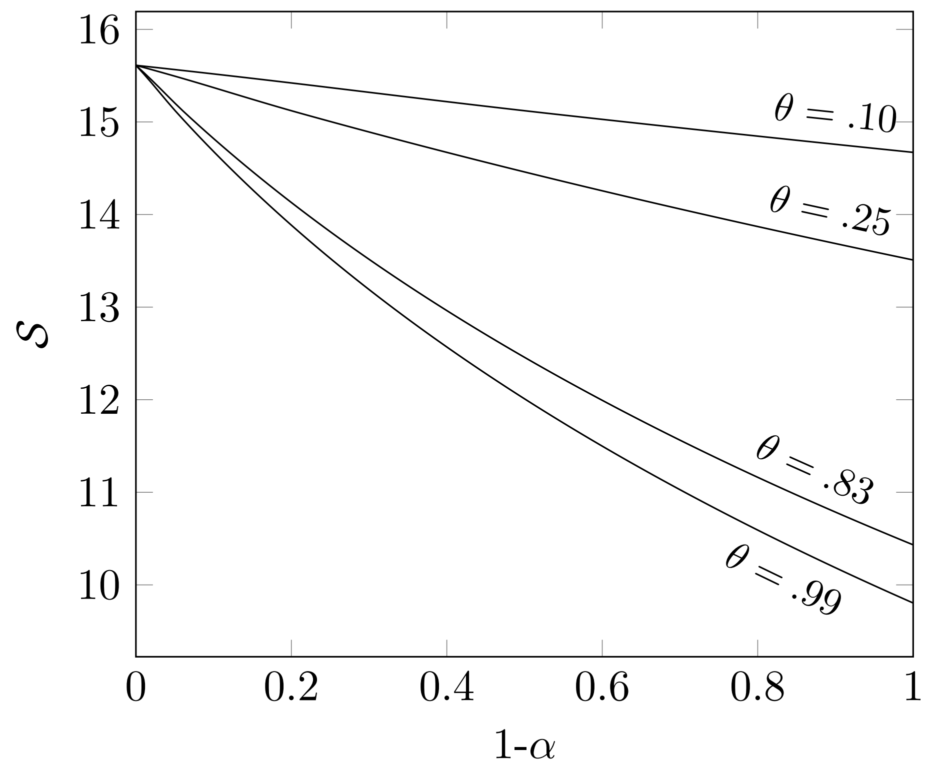


Figure 17: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1-\theta$, and access to credit, $1-\alpha$.

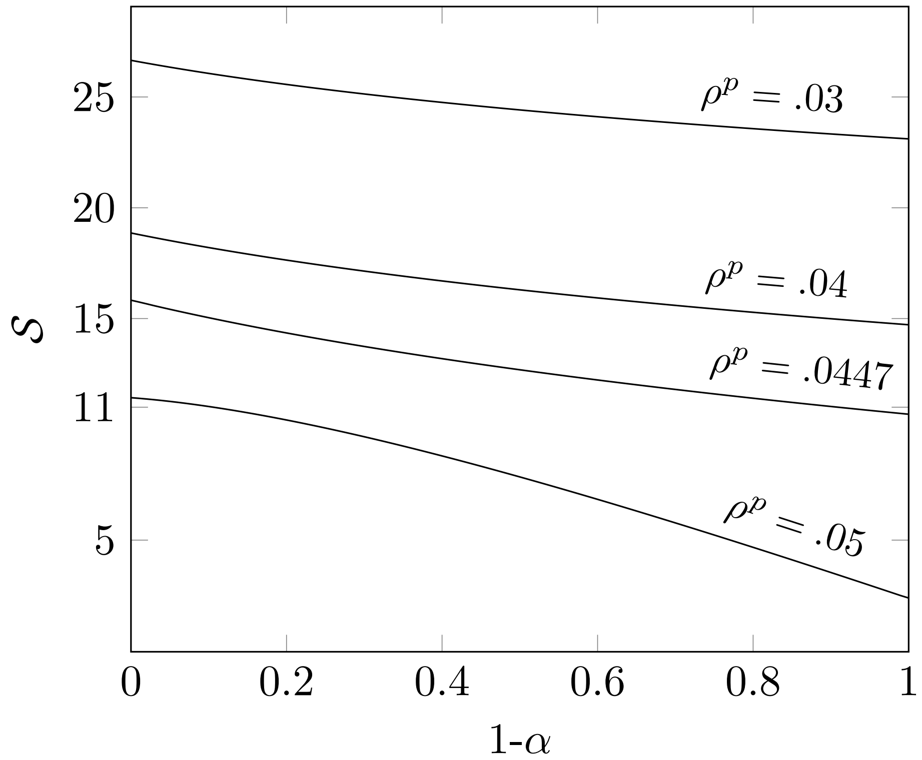
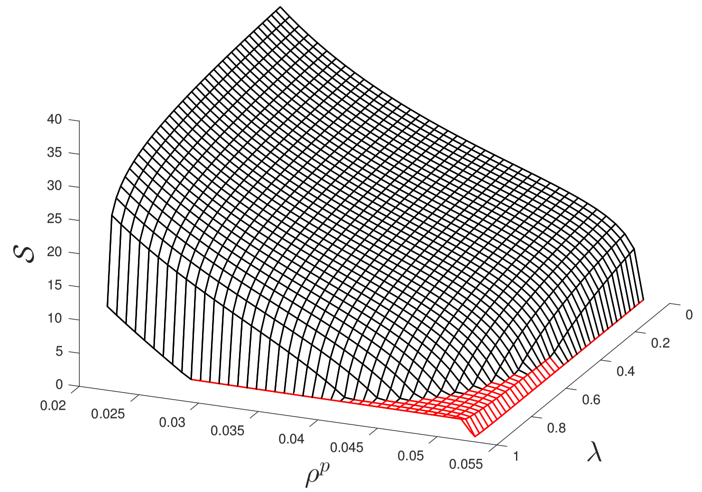


Figure 18: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1 - \alpha$.



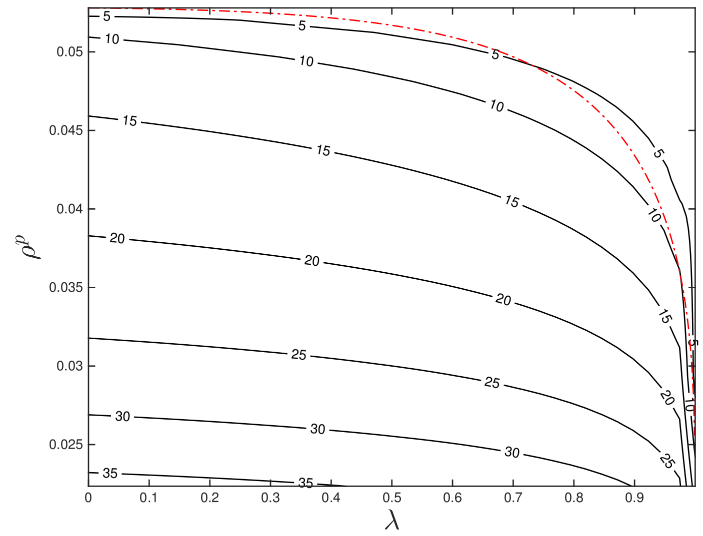


Figure 19: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

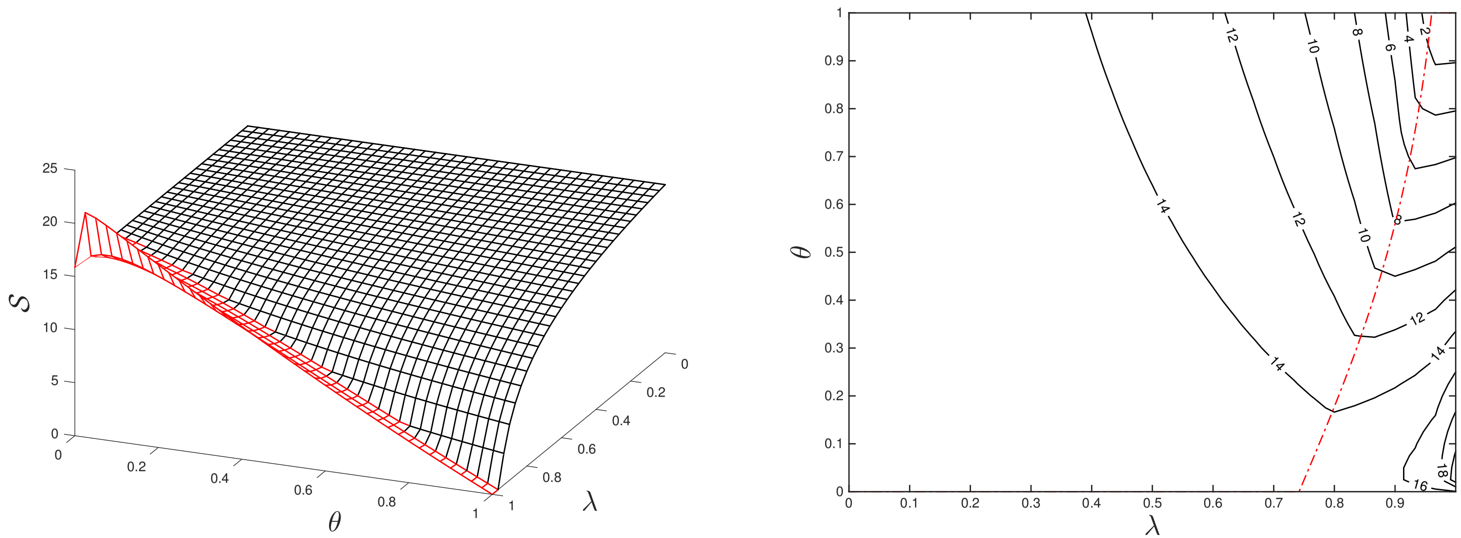


Figure 20: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).

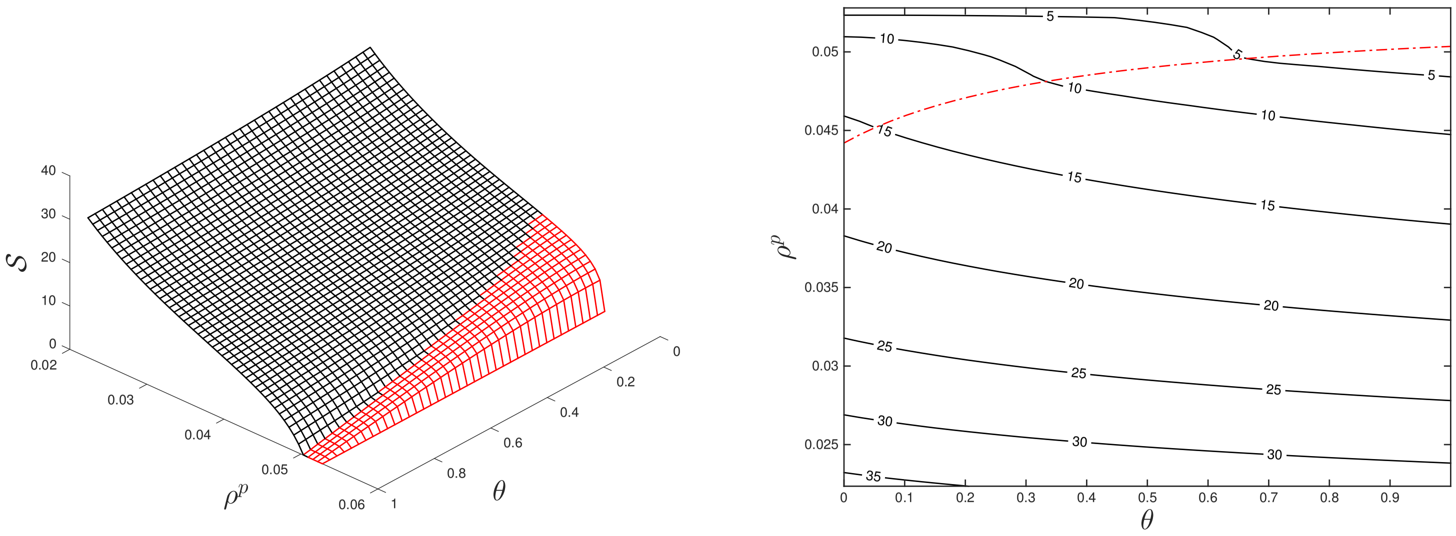


Figure 21: Calibration (AC2): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

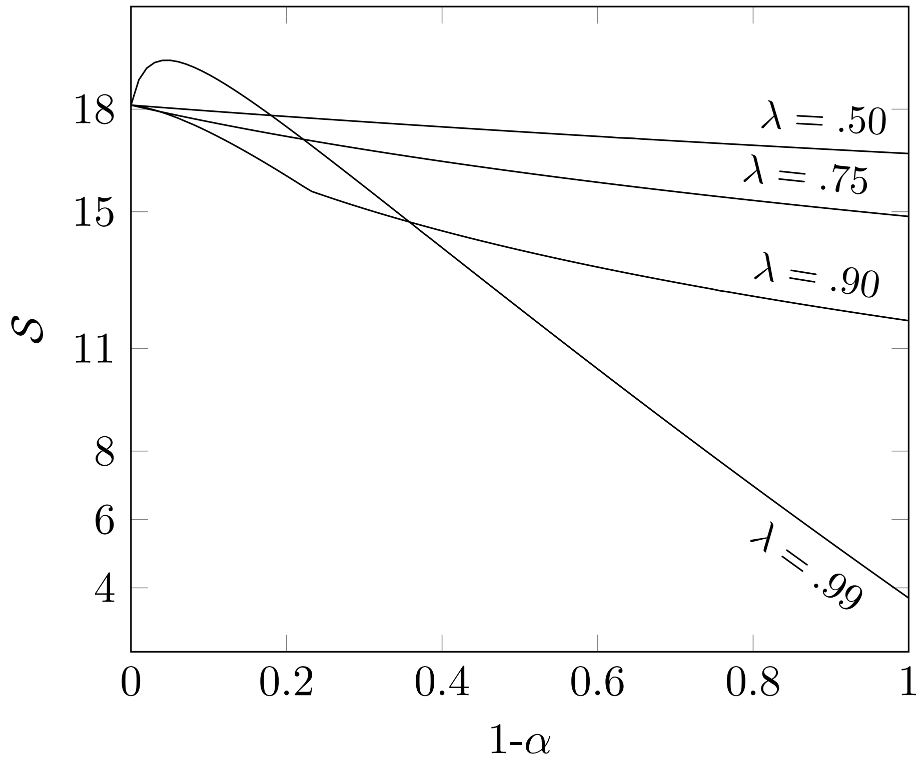


Figure 22: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different levels of leverage, λ, and access to credit, $1-\alpha$.

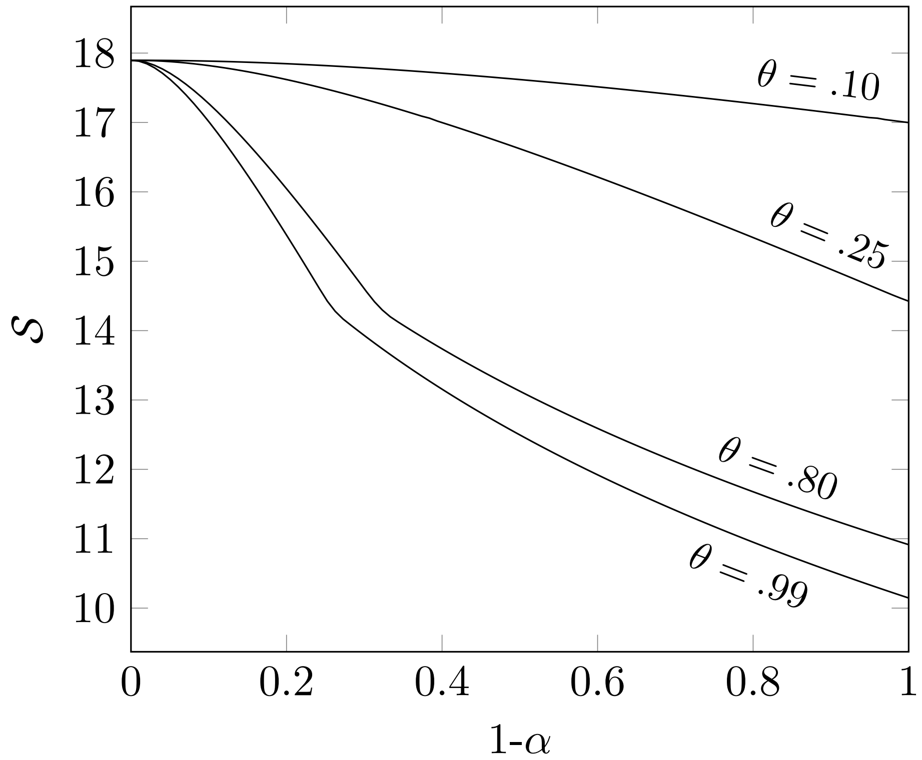


Figure 23: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different market power of brokers, $1-\theta$, and access to credit, $1-\alpha$.

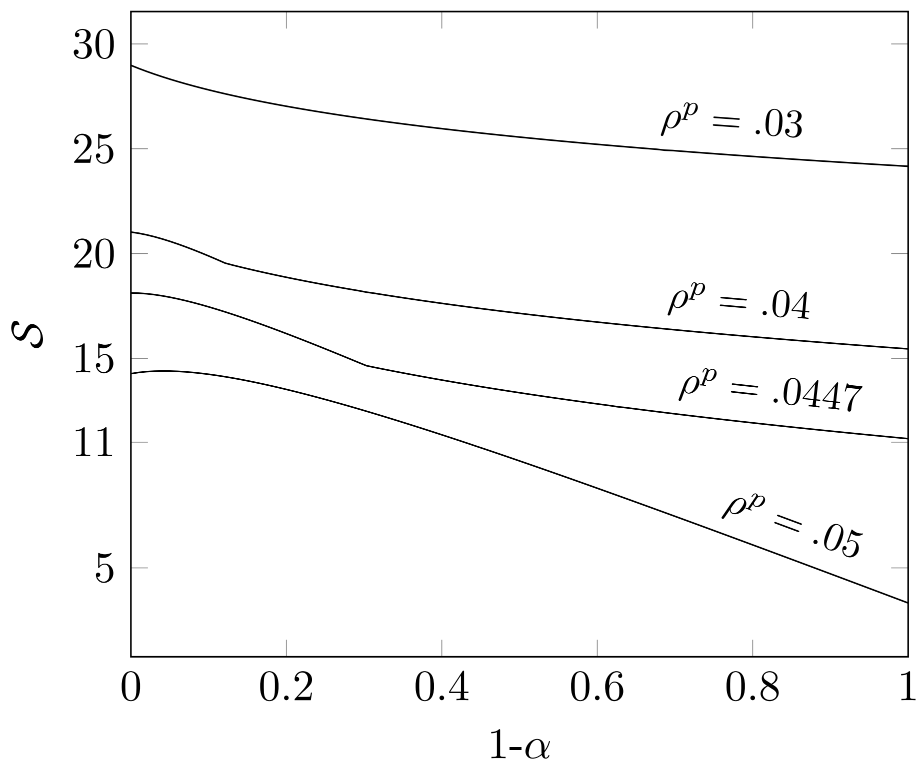


Figure 24: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate for economies with different monetary regimes, ρ^p, and access to credit, $1 - \alpha$.

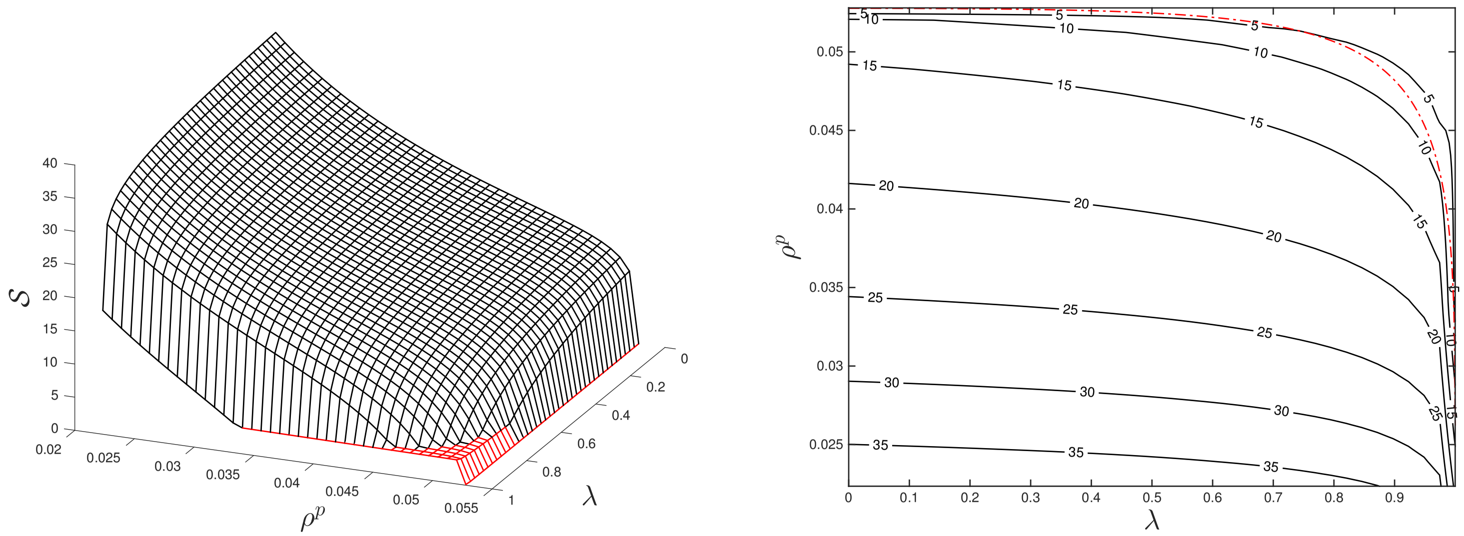


Figure 25: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).

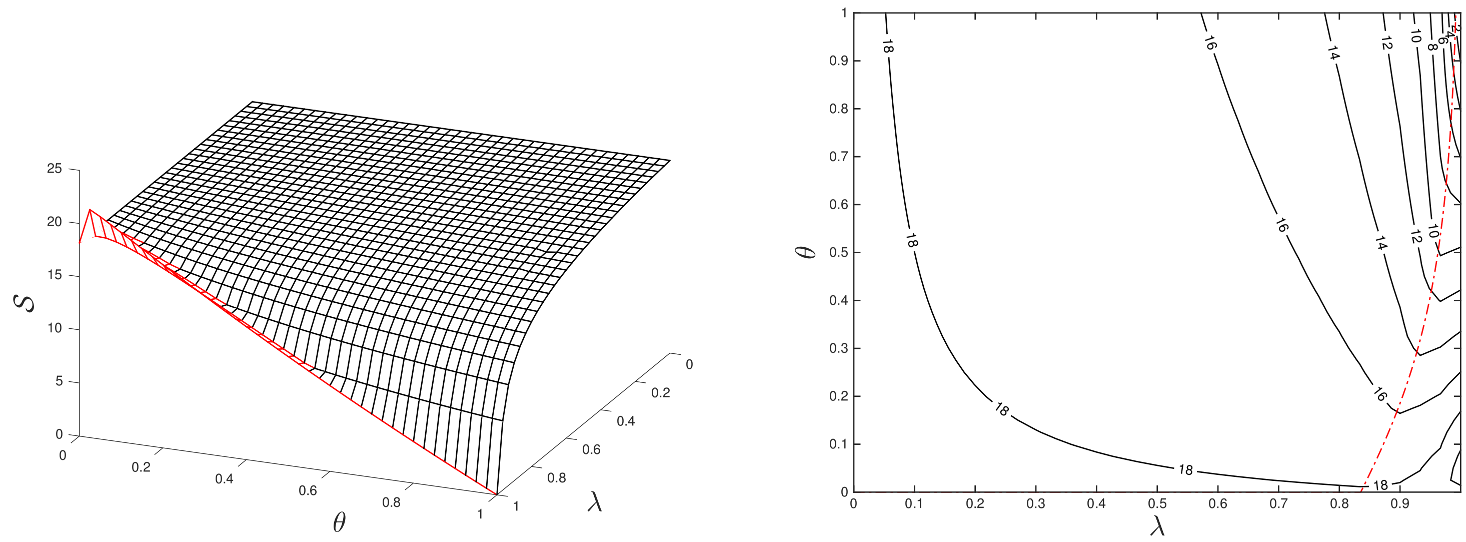


Figure 26: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of λ and θ in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie to the right of the dashed line).

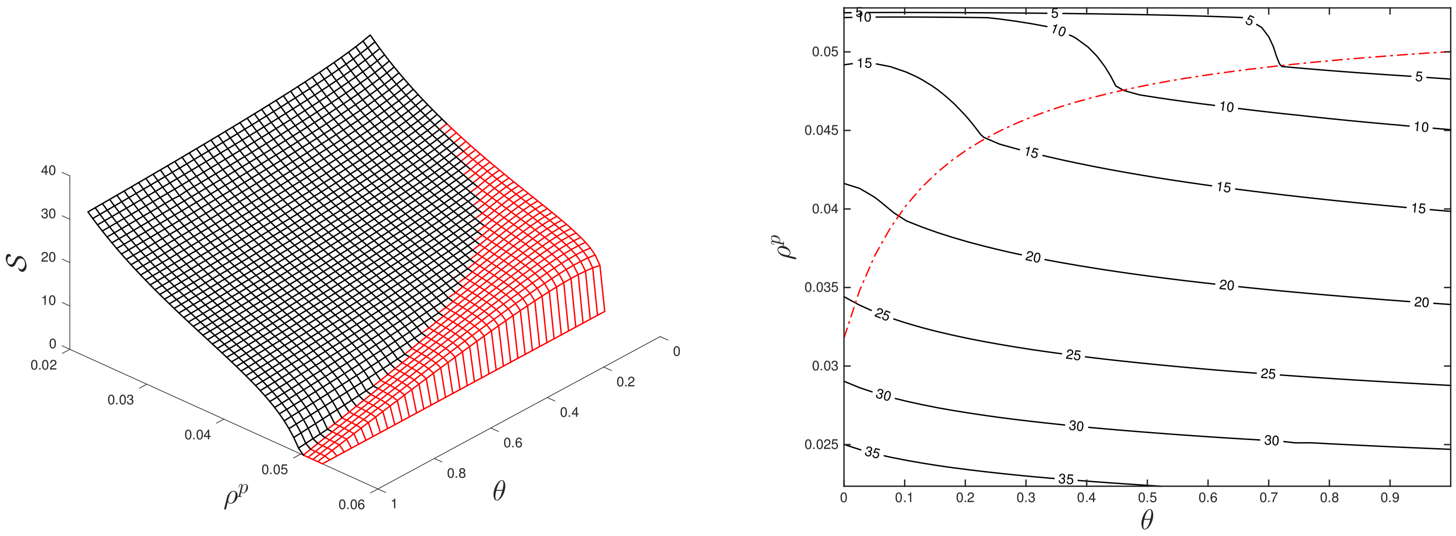


Figure 27: Calibration (AC3): Semi-elasticity of the asset price with respect to the nominal policy rate as functions of θ and ρ^p in limiting economies with $\alpha \to 0$. The right panel shows the level sets for $\mathcal{S}$ corresponding to the left panel (real money balances are zero for parametrizations that lie above the dashed line).