Stockholm University

# Essays on the Macroeconomics of Climate Change

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## Abstract

This thesis consists of three self-contained essays dealing with different macroeconomic aspects of climate change.

**Technological Trends and the Intertemporal Incentives for Fossil-Fuel use** analyzes how (the expectations about) the future developments of different kinds of technology affect the intertemporal incentives for fossil-fuel use. Given that fossil-fuel resources are finite, the decision of when to extract should be based on the value of fossil-fuel use at different points in time. This means that the expectations about the future state of technology matter for the extraction decisions made today. I find that improvements in (the expectations about) the future state of technologies for alternative-energy generation, energy efficiency and total factor productivity (TFP) all increase fossil-fuel use before the change takes place. The effect of changes in the efficiency of non-energy inputs is the reverse, while the effect of changes in fossil-fuel based energy technology is ambiguous. These conclusions are robust to a number of different possible variations of assumptions. Throughout this chapter, I emphasize the scarcity aspect of the fossil-fuel supply. This seems to be the crucial assumption. If fossil-fuel supply is, instead, mostly driven by extraction costs, some results may be reversed.

**The Role of the Nature of Damages** considers different ways in which climate change can be assumed to affect the economy (e.g., through various damages) and to what extent the choice of how to model these climate effects matters. In particular, I consider the choice of modeling climate impacts as affecting productivity, utility or the depreciation of capital.

I carry out my analysis in two different ways. Firstly, under some simplifying assumptions, I derive a formula for the optimal tax on fossil-fuel use. The optimal tax at each point in time can be written as a constant times current production, where the constant adds up the three different types of effects. Secondly, I use a two-period model with exogenous climate to analyze how the allocation of fossil-fuel use over time is affected by the effects of climate change. I consider two different cases for the fossil-fuel supply: an oil case, that emphasizes scarcity, and a coal case, that emphasizes extraction costs. I find that, for both the oil and coal cases, a decrease in second-period productivity and a worsening of the second-period climate state have the same qualitative effects on the allocation of fossil-fuel use while an increase in the depreciation of

capital has the opposite effect. The effects are also very different in the coal case compared to the oil case. I then ask whether these reactions to climate change will amplify or dampen climate change. I find that climate effects on productivity or utility will dampen climate change in the oil case and amplify it in the coal case. The opposite holds for effects on capital depreciation.

**Indirect Effects of Climate Change** investigates how direct effects of climate change in some countries have indirect effects on other countries going through changing world market prices of goods and financial instruments. When calculating the total effects of climate change these indirect effects must also be taken into account.

If climate change decreases the productivity of a country that is a net exporter of a good, the world market price will go up, decreasing the welfare in countries that are net importers of that good. Financial instruments can be used to insure against weather related uncertainty. The probability distribution of weather events is expected to change due to climate change. This means that the world market prices of financial instruments will change as the probability distribution of weather events changes. The indirect effects going through the price changes of assets will benefit or hurt countries depending on whether they are net buyers or net sellers of the assets.

Cost-efficient mitigation of climate change (reduction of emissions of greenhouse gases) requires reductions in all countries. The uneven distribution of the effects of climate change poses a problem for agreeing on mitigation efforts, especially since there seems to be a negative correlation between emissions of greenhouse gases and the vulnerability to climate change. Including the indirect effects gives a different distribution of total effects which can make it easier or more difficult to reach agreements depending on whether the indirect effects make the countries' interests more or less aligned. The net effects will depend on the relation between the direct effects and the trade patterns. I argue, based on a stylized two country example, that trade in goods will tend to make the countries' interests more aligned while trade in financial instruments will tend to make the countries' interests less aligned.

To Ulrika

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My joining the IIES coincided with the start-up of the Mistra-SWECIA programme on climate change. This gave me the opportunity to discover the field of the macroeconomics of climate change alongside some outstanding researchers: John Hassler, Per Krusell, Conny Olovsson, Torsten Persson and David von Below. It meant that I always had someone to talk to when I got stuck but also that I could observe how they approached a new research field. This gave me very valuable insights into the research process. Being part of the interdisciplinary Mistra-SWECIA programme also gave me a much broader understanding of the climate change issue.

During my time in the PhD program, I have had many great fellow students. I would especially like to mention three of them with whom I have had many inspiring discussions: David Yanagizawa Drott, Daniel Spiro and Gustav Engström.

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# Contents

# Chapter 1

# Introduction

This thesis consists of three self-contained essays on issues related to the macroeconomics of climate change. The first two chapters are relatively similar in terms of the questions asked and the models used to answer them. They both use neoclassical growth models where the world is treated as one large economy. They both consider issues of intertemporal incentives for fossil-fuel use and the use of taxation to correct for the externalities through climate change caused by the burning of fossil fuels. One might say that these two chapters deal with allocation over time. The third chapter, instead, uses models with many countries and considers how effects of climate change propagate between countries through market mechanisms. That is, it considers allocation across countries.

Climate change has become a topic of intense public debate in recent years. One contributing factor to this was the publication of the Stern Review (Stern, 2007). The basic mechanisms that are driving climate change have been known for a long time. More than a hundred years ago the increase in the global temperature following an increased concentration of greenhouse gases in the atmosphere was calculated fairly accurately. At that time, however, this was not necessarily considered a threat (for instance, the Swedish chemist and physicist Svante Arrhenius who was one of the pioneers thought, understandably, that a warmer climate might well be beneficial). Over time, the problems and risks associated with climate change have become more and more apparent.

The Intergovernmental Panel on Climate Change was created in 1988.[1] It has since then published four assessment reports and a fifth is scheduled to be published in 2013 and 2014. The fourth assessment report, published in 2007, received much attention and the organization, together with Al Gore, was awarded the 2007 Nobel Peace Prize.

Climate change is not a new topic within economic research either.

[1] See www.ipcc.ch.

Perhaps the best known economist working with these issues is William Nordhaus; he has studied the interaction between the climate and the economy since the 1970s. Nordhaus has also developed one of the most widely used family of tools that jointly model the economy and the climate: the DICE/RICE models.[2] While his importance for bringing together models of the climate and the economy is difficult to overestimate, these models (and most other so called IAMs, i.e., integrated assessment models) have a problem: they are highly complex and difficult to use for qualitative interpretation. One reason for this is that they consist of a large set of equations that can only be solved numerically.

The Mistra SWEdish research programme on Climate, Impacts and Adaptation (Mistra-SWECIA), which I have been a part of, was started in 2008. One of the main purposes of the macroeconomic modeling part of the programme was to approach the problem somewhat differently. The economic part of the models should be based on modern macroeconomic theory, making the models accessible to mainstream macroeconomists. The models should also be more transparent.

The work in this thesis very much reflects this aim for transparency. Rather than using large complex models, the chapters in this thesis explores qualitative issues using tractable models. I also think that the results derived in the thesis point to the value of this approach. When setting up an integrated assessment model, a number of assumptions must be explicitly or implicitly made. These assumptions can completely change the qualitative behavior of the model.

One important part of any climate-economy model is the supply of fossil fuels. Fossil-fuel resources are finite (at relevant time scales) and there is a cost of extracting them. An important question is which of these aspects of the resource is more important for extraction decisions. If the finiteness, or scarcity, of the resources is more important, comparing the value of fossil-fuel use at different points in time will be an important driver behind the intertemporal pattern of extraction. If, instead, the costs of extraction are more important, the extraction decisions will be more about weighing current extraction costs against the current value of fossil-fuel use at each point in time.

Another important issue is alternative-energy generation. Large reductions in fossil-fuel use without large reductions in material well-being will require a rapid increase in the use of energy generated by alternative sources. The way that the alternative-energy generation is modeled can have significant consequences for the behavior of IAMs. For example, if

[2] DICE and RICE stands for "Dynamic Integrated model of Climate and the Economy" and "Regional dynamic Integrated model of Climate and the Economy", respectively. See, e.g., Nordhaus & Boyer (2000) for a description of the models.

the production function is assumed to have some degree of complementarity between energy and other inputs the alternative-energy assumption becomes important. If the model abstracts from alternative energy, or if alternative energy is exogenously given, the complementarity between energy and other inputs translates into complementarity between fossil fuel and other inputs. If, instead, the capacity for generating alternative energy comes from use of inputs such as installed capital for energy generation, this implies something very different regarding the complementarity between fossil fuel and other inputs.

Furthermore, the functional forms for the production and utility functions must be specified. A relatively common assumption regarding the production function is that energy is combined with other inputs, such as labor and capital, according to a Cobb-Douglas production function. In both chapters 2 and 3, it can be seen that this assumption, especially if combined with the assumption that the utility function is logarithmic, significantly simplifies the analysis. It can, however, also be seen that these assumptions take away some mechanisms that would be present if more realistic assumptions were made.

It may not be a very surprising conclusion that the assumptions made when building a model affects the results that the model delivers. I would, however, argue that the different possibilities that I consider in this thesis lie within the span of model assumptions used and that conclusions are sometimes drawn that rely on the particular assumptions made. At the same time, the quantitative basis for making these assumptions is sometimes weak. Thus, while the analysis in this thesis often stops at the point where the consequences of making the different possible assumptions have been determined, this points to fruitful avenues for future research. Quantitative analysis of these possible choices is needed to find out what the "right" assumptions are and the quantitative consequences for model output such as optimal taxes on fossil-fuel use must be determined.

**Chapter 2, Technological Trends and the Intertemporal Incentives for Fossil-Fuel Use**, analyzes how (the expectations about) the future developments of different kinds of technology affect the intertemporal incentives for fossil-fuel use. Given that fossil-fuel resources are finite, the decision of when to extract should be based on the value of fossil-fuel use at different points in time. This means that the expectations about the future value of fossil-fuel use matters also for the extraction decisions made today. The future development of technology is an important determinant of this future value. The literature on the Green Paradox (see van der Werf and di Maria, 2011, for a survey of this literature) has recognized the importance of this aspect of the fossil-fuel supply for the effects of policies aimed at reducing the emissions of $CO_2$ from the burning of fossil fuels. What is found in this literature is that announced policies that reduce the future value of fossil-fuel use will tend to increase the current amount of fossil-fuel use and thereby potentially exacerbate the problem of climate change. The commonly discussed policies are announcements of higher future taxes on fossil-fuel use or investments that will increase the future supply of alternative energy.

The topic of this chapter is to consider how (the expectations about) the future developments of a wider range of technologies affect the intertemporal incentives for fossil-fuel use. The technology trends that I consider are technology for: alternative-energy generation, fossil-fuel based energy generation, energy savings, productivity of other (complementary) inputs, i.e., labor and sometimes capital, and total factor productivity (TFP). The analysis in this chapter is carried out using neoclassical models. I use these models to determine the effect of a future change in the state of each of the technologies on the path of fossil-fuel use. The general conclusion is that improvements in (the expectations about) the future state of technologies for alternative-energy generation, energy efficiency and TFP all increase fossil-fuel use before the change takes place. The effect of changes in the efficiency of non-energy inputs is the reverse, while the effect of changes in fossil-fuel based energy technology is ambiguous. These conclusions are robust to a number of different possible assumptions. Thus, the effects of changes in the future technology for alternative-energy generation and energy efficiency confirm the findings in the Green Paradox literature.

The analysis indicates that the joint effects of all technology trends should be considered rather than looking at one type of technology in isolation. In reality technology trends are the results of research. Increasing spending on one type of research will typically have effects also on the amount of research on other types technologies, e.g., through crowding out.

Throughout this chapter, I emphasize the scarcity aspect of the fossil-fuel supply. This seems to be the crucial assumption. If fossil-fuel supply is, instead, mostly driven by extraction costs, some results may be reversed.

**Chapter 3, The Role of the Nature of Damages**, considers different ways in which climate change can be assumed to affect the economy (e.g. through various damages) and to what extent the choice of how to model these climate effects matters. The most common way to introduce the effects of climate change into economic models is to assume that it affects productivity or utility. Some of the expected effects of climate change, e.g., storms and floods, rather destroy the capital stock. Modeling the effects of a climate change as increased depreciation of capital therefore seems plausible. In this chapter I consider to what extent it matters whether climate is assumed to affect productivity, utility or the depreciation of capital.

I carry out my analysis in two different ways. Firstly, under some simplifying assumptions, I derive a formula for the optimal tax on fossil-fuel use. The optimal tax at each point in time can be written as a constant times current production, where the constant adds up the three different effects that climate change has on the economy. Golosov et al. (2011) derive a similar formula for the optimal tax when considering only climate change effects on productivity. The formula derived in chapter 3 can therefore be seen as a generalization of that formula. The assumptions I make in order to derive the formula are also similar.

Secondly, I use a two-period model with exogenous climate to analyze how the allocation of fossil-fuel use over time is affected by the effects of climate change. I consider two different cases for the fossil-fuel supply: an oil case, where the resources are finite but I abstract from extraction costs, and a coal case, where I abstract from the finiteness of the resource but extraction requires the use of inputs. I find that, for both the oil and coal cases, a decrease in second-period productivity and a worsening of the second-period climate state have the same qualitative effects on the allocation of fossil-fuel use while an increase in the depreciation of capital has the opposite effect. The effects are also very different in the coal case compared to the oil case.

In the second part of this chapter, I treat climate as exogenous. The derived effects are still indicative of how a decentralized equilibrium would respond to expected climate change. I then ask whether these reactions to climate change will amplify or dampen climate change. Answering this question requires assumptions about how to best represent the effects of climate change in a two-period model and what constitutes amplification of climate change in the oil and coal cases, respectively. Under the interpretation I choose, climate effects on productivity or utility will dampen climate change in the oil case and amplify it in the coal case. Conversely, climate effects on the depreciation of capital will amplify climate change in the oil case, at least if the supply of alternative energy is exogenously given, but dampen it in the coal case.

**Chapter 4, Indirect Effects of Climate Change**, investigates how direct effects of climate change in some countries have indirect effects on other countries going through changing world market prices of goods and financial instruments. The direct effects of climate change are expected to differ a great deal across different countries. However, since the economies of countries are interconnected in various ways the direct effects will be propagated between countries through market mechanisms. This means that when calculating the total effects of climate change these indirect effects must also be taken into account.

In this chapter I consider two such channels: trade in goods and trade in financial instruments. For both of these channels the indirect effects go through changing world-market prices of goods and financial instruments. If climate change decreases the productivity of a country that is a net exporter of a good, the world market price will go up, decreasing the welfare in countries that are net importers of that good. Weather events cause uncertainty. Financial instruments can be used to decrease this uncertainty by offering insurance against bad outcomes. The probability distribution of weather events is expected to change due to climate change. This means that the world market prices of financial instruments will change as the probability distribution of weather events changes. The indirect effects going through the price changes of assets will benefit or hurt countries depending on whether they are net buyers or net sellers of the assets.

Climate change depends primarily on total global emissions of greenhouse gases while the geographical source of the emissions are largely irrelevant. This means that cost-efficient mitigation of climate change (reduction of emissions of greenhouse gases) requires reductions in all countries. The uneven distribution of the effects of climate change poses a problem for efficient mitigation since countries willingness to participate in mitigation efforts can be expected to be closely related to the costs from climate change they are expected to suffer. This is made worse by the fact that there seems to be a negative correlation between emissions of greenhouse gases and the vulnerability to climate change. Since the indirect effects of climate change will give a different distribution of the total effects compared to the distribution of direct effects, these indirect effects can make it easier or more difficult to reach agreements about mitigation efforts depending on whether the indirect effects make the countries' interests more or less aligned. The net effects will depend on the relation between the direct effects and the trade patterns. I argue, based on a stylized two country example, that trade in goods will tend to make the countries' interests more aligned while trade in financial instruments will tend to make the countries' interests less aligned.

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# Chapter 2

# Technological Trends and the Intertemporal Incentives for Fossil-Fuel Use

## 2.1 Introduction

In order to avoid risks of serious negative effects of climate change, large reductions of emissions of greenhouse gases are discussed. In order to accomplish this without large reductions in economic growth, the use of fossil fuel must rapidly be replaced by energy from alternative sources. However, the finiteness of the fossil-fuel resources introduces a kind of intertemporal incentive for extraction that can give unexpected side effects from investments in alternative energy technology. This is known as the green paradox. In this chapter I will extend previous research to consider more generally how technological trends affect the intertemporal incentives for fossil-fuel extraction.

The Copenhagen accord states that the countries should aim to fulfill the two degree target, meaning that the global mean temperature should not be allowed to increase more than two degrees above pre-industrial levels. This requires very large reductions of the emissions of greenhouse gases, of which $CO_2$, from the burning of fossil fuels, is one of the most important ones. As an example, the European commission's roadmap for moving to a competitive low carbon economy in 2050 (European Commission 2011) says that the developed countries should reduce emissions of greenhouse gases by 80-95% by 2050 in order to reach the two degree target. At the same time it seems that, at least in the short run, there is low substitutability between energy and other inputs (see, e.g., Hassler et al. 2011). This implies that, in order to come anywhere near the two degree target, without seriously hurting economic activity, technological change is needed.

Fossil fuels are, on relevant time scales, non-renewable and extracted from a finite supply. The consequences of this, for the extraction decisions of fossil-fuel extracting firms, was first analyzed by Hotelling (1931). Forward looking and profit maximizing fossil-fuel resource owners should extract in such a way that the marginal (discounted) profits from extraction is the same in all periods where extraction is positive. This means that any changes in the future profitability of fossil-fuel extraction should affect the extraction decisions already today. This implies that any decrease in the future profitability of fossil-fuel extraction should lead to increased fossil-fuel extraction in the short run and that announcements about policies aimed at reducing emissions of greenhouse gases in the future can lead to increased emissions in the short run. Sinn (2008) considered such effects and coined the term Green Paradox. For the case of an increasing tax on fossil-fuel use, Sinclair (1992) found that this increases emissions in the short run. In the terminology of Gerlagh (2011), the Weak Green Paradox refers to a situation where changes in expectations about future taxation, or improvements in the future state of alternative-energy technology, generated by an ambition to reduce emissions of green house gases, counter-productively increase emissions in the short run. In this chapter I will mainly consider effects similar to the Weak Green Paradox. The Strong Green Paradox refers to a situation where, over the long run, the effects of climate change become worse as a consequence of regulation aimed at reducing climate change. A number of papers, for example Gerlagh (2011) and van der Ploeg and Withagen (2012), have further investigated these mechanisms. Van der Werf and Di Maria (2011) provide an overview of this recent literature.

Thus the literature on the green paradox is concerned with how changes in expectations about future development of alternative-energy technology, or future taxation, affect fossil-fuel use. The result most relevant for this chapter is that an improvement in the future availability of alternative-energy increases fossil-fuel use in the short run. Motivated by these studies, the present chapter addresses a broad question: what are the effects of the path of technological development on fossil-fuel use? I also extend the analysis to include many different kinds of technology. The technology trends that I will study are technology for alternative-energy generation, fossil-fuel based energy generation, energy savings, productivity of other (complementary) inputs, i.e., labor and sometimes capital, and general TFP.

I carry out the analysis in the framework of a neoclassical model with fossil fuel as a non renewable resource. In some parts of the chapter, fossil-fuel use causes climate change that affects future productivity. The model can be seen as a variation of the model used by Dasgupta and Heal (1974). Compared to that model, I add climate change caused by the use of fossil fuels as well as the use of alternative-energy sources.

By specifying a general, nested, CES production function, and a CES utility function, I can see how the effects of changes in different technology trends depend on the parameters of the production and utility functions. There are two parameters in the production function. One parameter determines the degree of complementarity between energy and other inputs (where the other inputs are labor and sometimes also capital) and the other parameter determines the degree of complementarity (or, rather, substitutability) between different energy sources.

Starting with a model without capital or any climate change related externalities, a set of reasonable assumptions about the values of the parameters allows me to unambiguously determine the effect of changes in all the considered technology trends except for the technology for fossil-fuel based energy generation. For the other technology factors, an increase in the future state of TFP, energy-saving technology and alternative-energy technology increases fossil-fuel use in the short run. An increase in the future state of the productivity of the complementary inputs decreases fossil-fuel use in the short run. The assumptions that allow me to derive these results are that the CES utility function has at least logarithmic curvature, that there is a significant degree of complementarity between energy and other inputs and that different energy sources are close substitutes. This analysis is carried out in section 2.2. The rest of the chapter then considers various extensions and investigates the robustness of the basic results to these extensions. It turns out that the qualitative results are quite robust to these other aspects. The only case in which the basic results do not hold is when alternative-energy generation uses an endogenously determined input that is extracted using labor. The effect of a change in the labor intensive technology is then ambiguous. In summary, considering the effects of changes in the future state of the technology for alternative-energy generation and energy-saving technology, I obtain a rather general Weak Green Paradox.

Throughout, I treat the technology trends as exogenous. In reality, technological development is driven by research activity. To the extent that research uses some scarce resources, increased research on one type of technology will tend to crowd out other types of research. The results derived in this chapter implies that it matters what kind of research is crowded out. The mechanism of the green paradox will be reinforced or weakened by this crowding out depending on what type of other research is crowded out.

When discussing how fossil-fuel resource owners react to changes in expectations about future technology this implies an interpretation in terms of a market outcome. However, if this reaction is also the optimal reaction, this should not be much of a problem. I therefore, when they do not coincide, solve for both the market outcome and the planning solution. I find that the change in the planner solution goes in the same direction as in the decentralized outcome. This implies that it is interesting to look more at how optimal taxation and the welfare gains from taxation change if the future state of technology changes. I find that if the change in the future state of technology is such that fossil-fuel use increases in the short run, then in the short run the tax rate that can be used to implement the optimal solution in a competitive equilibrium with taxation increases too. Regarding the welfare effects of taxation, I demonstrate that these could go either way.

In parts of the chapter, I abstract from capital accumulation to simplify the analysis. I check the robustness to including capital in two different ways. In section 2.3, I consider a two-period model with capital and relatively general forms of the utility and production functions. I also show, in section 2.5, that if capital combines with other inputs as in a Cobb-Douglas production function, if utility is logarithmic and if capital depreciates fully between periods, then almost all the derived results will apply equally well to a model with capital. Throughout the chapter, I will assume that fossil fuel is costlessly extracted from a given total supply. This is a strong assumption and I will discuss it further in section 2.7.

The rest of the chapter is organized as follows. I start in section 2.2 by setting up a model without capital and without externalities. Initially, production in each period just depends on fossil-fuel use and a set of exogenously given variables. The exogenous variables are then specified as other inputs and technology factors. This formulation allows me to investigate how the results depend on particular parameters in the production and utility functions. After that, in section 2.3, I look at a two-period model with capital but without externalities. Then, in section 2.4, I introduce externalities in the form of climate change. With externalities in the model, I need to distinguish between a decentralized equilibrium, section 2.4.2, and the planner solution, section 2.4.3. With externalities, I can also discuss optimal taxation, in section 2.4.4, and the welfare gains from taxation, in section 2.4.5. After that, in section 2.5, I show that for the special case where energy and the other inputs are combined into final goods according to a Cobb-Douglas production function, utility is logarithmic and capital depreciates fully, the solutions simplify a lot and almost all the result derived for models without capital hold also with capital. In all previous sections, the supply of alternative energy has been assumed to depend on the level of technology for alternative-energy generation and an exogenously given amount of an alternative-energy input, implying that improved technology immediately transforms into increased use of alternative energy. In section 2.6 I, instead, assume that the alternative-energy input is endogenously determined based on the resources required to provide it. Finally, the chapter is concluded with a discussion of the derived results and the assumptions made.

## 2.2 Model without capital and externalities

This section contains much of the basic intuition underlying the results of this chapter. I will start by setting up a model where the only endogenously determined variables are the amounts of fossil-fuel use in each period. Initially, production will depend on fossil-fuel use and a set of abstract, exogenously given, variables. I will consider how varying the exogenously given variables affects the equilibrium allocation of fossil-fuel use. This demonstrates the basic mechanisms involved that affect the incentives for intertemporal allocation of fossil-fuel use. After that, I specify a specific production and utility function so that I can determine the effects of changing particular technology factors and exogenously given inputs.

### 2.2.1 Model setup

Fossil fuel is costlessly extracted from a fixed supply. Let the amount of fuel burned in period $t$ be $B_t$ and the amount of fuel left in the ground at the beginning of period $t$ be $Q_t$.

The constraint on the total available amount of fossil fuel can then be written

$$\sum_{t=0}^{\infty} B_t \leq Q_0. \tag{2.1}$$

Production in a period depends on the amount of fossil fuel used and on a set of exogenously given variables $\Gamma$

$$Y = F(B; \Gamma).$$

The production function is assumed to have the properties

$$\frac{\partial F}{\partial B} > 0 \text{ and } \frac{\partial^2 F}{\partial B^2} < 0. \tag{2.2}$$

If the production function also fulfills the condition

$$\lim_{B \to 0^+} \frac{\partial F}{\partial B} = \infty, \tag{2.3}$$

then fossil-fuel use will be strictly positive in each period. I will also assume that the variables are defined so that production depends positively on each variable in $\Gamma$.

Consumption is equal to production in each period $C_t = Y_t$. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t U(C_t),$$

where $\beta$ is the discount factor and the period utility function is assumed to have the properties

$$U'(C) > 0 \text{ and } U''(C) < 0. \tag{2.4}$$

### 2.2.2 Equilibrium

Without externalities in the form of climate change, the planner solution will coincide with the competitive equilibrium and I will therefore solve the planner problem.

The planner problem is to maximize utility given the constraint on the total amount of available fossil fuel:

$$\max_{\{B_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(Y_t) \text{ s.t. } B_t \geq 0 \, \forall t \text{ and } \sum_{t=0}^{\infty} B_t \leq Q_0.$$

The Lagrangian of this problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(Y_t) + \lambda \left[ Q_0 - \sum_{t=0}^{\infty} B_t \right] + \sum_{t=0}^{\infty} \mu_t B_t.$$

The first order condition with respect to $B_t$ is

$$\beta^t U'(C_t) \frac{\partial Y_t}{\partial B_t} = \lambda - \mu_t,$$

where $\lambda > 0$ if the constraint on total available fossil fuel binds and $\mu_t > 0$ if $B_t \geq 0$ binds.

Without externalities, and given assumption (2.2), the constraint on the total supply of fossil fuel will always bind and $\lambda > 0$. Assuming that $B_T > 0$, for some $T$, the equilibrium condition can be written

$$\beta^t U'(C_t) \frac{\partial Y_t}{\partial B_t} \leq U'(C_T) \frac{\partial Y_T}{\partial B_T} \text{ and } \sum_{t=0}^{\infty} B_t = Q_0$$

with equality whenever $B_t > 0$. If the production function fulfills assumption (2.3), $B_t > 0$ for all $t$.

In essence, what the solution does is that it equalizes the marginal value of fossil-fuel use over time.

Define

$$H(B;\Gamma) = U'(Y)\frac{\partial Y}{\partial B}. \tag{2.5}$$

That is, $H$ is the marginal value, in terms of utility, of fossil-fuel use. With this definition, assuming that $B_T > 0$, the equilibrium is characterized by the two conditions:

$$\beta^t H(B_t;\Gamma_t) = \beta^T H(B_T;\Gamma_T) \text{ for all } t \text{ such that } B_t > 0 \tag{2.6}$$

and

$$\sum_{t=0}^{\infty} B_t = Q_0. \tag{2.7}$$

### 2.2.3 Changes in the exogenous variables

I will now show how changes in the exogenously given variables affect the equilibrium allocation of fossil-fuel use. If $\Gamma_T$ changes, for $T$ such that $B_T > 0$, this will change the marginal value of fossil-fuel use in that period. Since the equilibrium allocation requires that the marginal value of fossil-fuel use be the same in all periods, and since fossil-fuel use is the only endogenously determined variable, fossil-fuel use will change in reaction to the change in $\Gamma_T$ in order to equalize the marginal value.

The effect on the marginal value of fossil-fuel use of changes in fossil-fuel use is

$$\frac{\partial H}{\partial B} = U''(Y)\left(\frac{\partial Y}{\partial B}\right)^2 + U'(Y)\frac{\partial^2 Y}{\partial B^2}.$$

Under assumptions (2.2) and (2.4)

$$\frac{\partial H}{\partial B} < 0. \tag{2.8}$$

Thus everything else equal, an increase in the fossil-fuel use in a period decreases the marginal value of using fossil fuel in that period.

Changes in $\Gamma_T$ change the relative values of using fossil fuel in period $T$ compared to the value of fossil-fuel use in other periods. If the change in $\Gamma_T$ increases the value of fossil-fuel use in period $T$, this should lead to a redistribution of fossil-fuel use toward period $T$. Similarly, a change in $\Gamma_T$ that decreases the value of fossil-fuel use in period $T$ should lead to a redistribution of fossil-fuel use away from period $T$. This is formalized in the following proposition.

**Proposition 2.1.** *Assume that the sequence* $\{\Gamma_t\}_{t=0}^{\infty}$ *induces the sequence* $\{B_t\}_{t=0}^{\infty}$ *of fossil-fuel use. Consider two periods* $t$ *and* $T$ *such* *that $T \neq t$, $B_t > 0$ and $B_T > 0$ and a change in $X_T$, defined as one of the variables in $\Gamma_T$. Then*

$$Sgn\left(\frac{dB_T}{dX_T}\right) = Sgn\left(\frac{\partial H_T}{\partial X_T}\right) \text{ and } Sgn\left(\frac{dB_t}{dX_T}\right) = Sgn\left(-\frac{\partial H_T}{\partial X_T}\right).$$

*Proof.* In the induced outcome, (2.6) holds in all periods where there is strictly positive fossil-fuel use. If the variable $X_T$ is varied, the condition is no longer fulfilled in that period if the fossil-fuel use is unchanged.

If $\frac{\partial H_T}{\partial X_T} > 0$, (2.8) implies that $B_T$ has to increase to still satisfy the equilibrium condition. This would mean that the constraint on the total supply of fossil fuel is violated. Using (2.8) again, changing the fossil-fuel use in one period, while keeping all variables constant and while maintaining the equilibrium condition (2.6) means changing the fossil-fuel use in all periods in the same direction. This means that the fossil-fuel use in all periods should be decreased until the constraint on total supply of fuels is fulfilled. So the net effect will be that fossil-fuel use is decreased in all periods with $B_t > 0$ except in period $T$ where the net effect will be to increase the fossil-fuel use.

The case $\frac{\partial H_T}{\partial X_T} < 0$ is the mirror image of the previous case.

□

The following corollary follows directly from this proposition:

**Corollary 2.1.** *Consider two sequences of parameters $\{\Gamma_t^I\}_{t=0}^{\infty}$ and $\{\Gamma_t^{II}\}_{t=0}^{\infty}$ with corresponding induced fossil-fuel use $\{B_t^I\}_{t=0}^{\infty}$ and $\{B_t^{II}\}_{t=0}^{\infty}$ respectively. Assuming that $\Gamma_t^I = \Gamma_t^{II}$ for all $t < T$ and that for $t \geq T$, $H(B_t^I; \Gamma_t^{II}) \leq H(B_t^I; \Gamma_t^I)$, then $B_t^I \geq B_t^{II}$ for all $t < T$ (and vice versa).*

*Proof.* Follows from proposition 2.1 □

Thus, if the expectations in period 0 about the future change in such a way that, from period $T$ and onwards, the value of using fossil fuel will decrease, there will be increased fossil-fuel use in all periods before $T$. If the change in expectations is such that the future value of using fossil fuel will increase, fossil-fuel use will decrease in all periods before $T$. Here the change in the value of fossil-fuel use comes from changes in the exogenous variables in the production function. In a decentralized equilibrium with taxation, credible announcements about the future taxes on fossil-fuel use will have a very similar effect since it affects the relative profitability of extracting fossil fuel in different time periods.

Note that corollary 2.1 is not sufficient for concluding how $B_t$ will change in individual periods for $t \geq T$. This is because the changes in driving variables in that period in the other periods will tend to move fossil-fuel use in opposite directions.

The proposition and corollary describe the effects of changes in $X_T$ (which is one of the variables in $\Gamma_T$) through how this change affects the marginal value of fossil-fuel use, $H_T$, in period $T$. The effect of changes in $X_T$ on $H_T$ can be divided into two separate effects according to the following derivative:

$$\frac{\partial H}{\partial X} = \frac{\partial}{\partial X}\left[U'(Y)\frac{\partial Y}{\partial B}\right] = U''(Y)\frac{\partial Y}{\partial X}\frac{\partial Y}{\partial B} + U'(Y)\frac{\partial^2 Y}{\partial X \partial B}. \tag{2.9}$$

The first term captures that the change in $X$ affects production directly and thereby affects the marginal value of consumption. Since, by assumption, $U''(Y) < 0$, $\frac{\partial Y}{\partial X} > 0$ and $\frac{\partial Y}{\partial B} > 0$, this effect is always negative, capturing that increased consumption decreases the marginal value of consumption. The second term captures that the change in $X$ also can have an effect on the marginal productivity of fossil fuel.

From this discussion it follows that if an increase in $X$ decreases the marginal product of fossil fuel, then it unambiguously decreases the marginal value of using fossil fuel in that period. If, instead, an increase in $X$ increases the marginal product of fossil fuel, the total effect is ambiguous and the sign depends on the relative strength of the effects.

In sum, the conclusion is that changes in $X_T$ that increase the value of fossil-fuel use in period $T$ will lead to an increase in fossil-fuel use in that period and a decrease in fossil-fuel use in all other periods. Changes in $X_T$ that decreases the value of fossil-fuel use will have the opposite effect. The effects of a change in $X_t$ on the value of fossil-fuel use in period $T$ is given by (2.9).

### 2.2.4 Interpretation of $\Gamma$

In order to say something more concrete about the effects of changing the future state of different technologies, I will now be specific about what the variables in $\Gamma$ are and what the production and utility functions look like.

There are three inputs to production. These are labor, $L$, which is assumed to be exogenously given, fossil fuel and an alternative-energy input, $S$, which also is exogenously given.

In addition to this, there are five technology factors:

1. $A_Y$ is general TFP
2. $A_L$ is labor-intensive technology
3. $A_E$ is general energy-saving technology
4. $A_B$ is fossil-fuel based technology

5. $A_S$ is alternative-energy technology

The technology factors are also assumed to develop exogenously.

Under these assumptions, the only variable that is endogenously determined is fossil-fuel use.

Define the vector of variables

$$\Gamma = (L, S, A_Y, A_L, A_E, A_B, A_S).$$

The effects of changing the different variables in $\Gamma$ will depend on the shape of the utility and production functions. Assume the utility function

$$U(C) = \frac{C^{1-\theta} - 1}{1 - \theta}, \tag{2.10}$$

a functional form that is needed for generating outcomes with balanced growth, and the production function

$$F(B; \Gamma) = A_Y \left[ \gamma_L \left( A_L L \right)^{\frac{\sigma_Y - 1}{\sigma_Y}} + \gamma_E \left( A_E Y_E \right)^{\frac{\sigma_Y - 1}{\sigma_Y}} \right]^{\frac{\sigma_Y}{\sigma_Y - 1}}, \tag{2.11}$$

where

$$Y_E = \left[ \gamma_B \left( A_B B \right)^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S \left( A_S S \right)^{\frac{\sigma_E - 1}{\sigma_E}} \right]^{\frac{\sigma_E}{\sigma_E - 1}} \tag{2.12}$$

is a composite energy good that is produced from fossil fuel and the alternative-energy source.

Note that there is one more technology factor here than is strictly necessary. Any combination of technology factors could be achieved with a smaller set of technology factors. For example, any combination of $A_B$, $A_E$ and $A_S$ can be achieved by setting any one of them equal to 1 and then adjusting the other two accordingly. There are at least two reasons for maintaining this redundancy. Firstly, each of the technology factors translates into measures that are commonly referred to: there are frequent discussions about energy efficiency, $A_E$ as well as technologies for different kinds of energy generation $A_B$ and $A_S$. Secondly, looking at the different technology factors, the effect of an arbitrary combination of changes in $A_B$ and $A_S$ will sometimes be ambiguous while the sign of the effect of the particular combination of changes in $A_B$ and $A_S$ that corresponds to a change in $A_E$ is unambiguously determined by the assumptions made below.

The effects will also depend on the parameters in the utility and production functions. In (2.11), $\sigma_Y$ gives the substitutability between energy and other inputs (here labor) while in (2.12), $\sigma_E$ gives the substitutability between the different energy sources. It seems reasonable that energy and other inputs are not very close substitutes while different energy sources are close substitutes. This leads to the following assumptions about parameter values:

$$\sigma_Y \leq 1 < \sigma_E. \tag{2.13}$$

Note that for a finite $\sigma_E$, this production function fulfills assumption (2.3) and fossil-fuel use will be strictly positive in all periods. In the limit as $\sigma_E \to \infty$, the different energy sources become perfect substitutes and fossil-fuel use will typically only be positive in a finite number of periods.

For the utility function, Layard et al. (2008) find that $\theta$ lies between 1.2 and 1.3. I will assume here that

$$\theta \geq 1. \tag{2.14}$$

Furthermore the following assumption will be made:

$$\frac{1}{\sigma_Y} \geq \theta. \tag{2.15}$$

This assumption says that the complementarity between energy and other inputs is strong (at least in relation to the curvature of the utility function). Hassler et al. (2011) estimate the elasticity between energy and a Cobb-Douglas composite of capital and labor to be about 0.005. One can expect that the elasticity is larger the larger the time period but assuming (2.15) still seems reasonable.

To simplify the notation a bit, let

$$\begin{aligned} G_L &= \gamma_L \left(A_L L\right)^{\frac{\sigma_Y - 1}{\sigma_Y}}, G_E = \gamma_E \left(A_E Y_E\right)^{\frac{\sigma_Y - 1}{\sigma_Y}}, \\ G_B &= \gamma_B \left(A_B B\right)^{\frac{\sigma_E - 1}{\sigma_E}}, G_S = \gamma_S \left(A_S S\right)^{\frac{\sigma_E - 1}{\sigma_E}}. \end{aligned} \tag{2.16}$$

Using the functional form of the utility function (2.10), the derivative of $H$ with respect to parameter $X$ (again defined as one of the variables in $\Gamma$) is

$$\frac{\partial H}{\partial X} = U''(C) F_X F_B + U'(C) F_{XB} = U'(Y) \left[ F_{XB} - \theta \frac{F_X F_B}{Y} \right], \tag{2.17}$$

where the subscripts to $F$ refers to partial derivatives.

Since $U'(C) > 0$, this expression will have the same sign as the last parenthesis. For qualitative results regarding increases or decreases in fossil-fuel use it is the sign that is important. Using (2.11) and (2.12), the expression (2.17) can be calculated for the different variables $X$. In

| wrt | sign of (2.17) |
|---|---|
| $A_Y$ | $1-\theta$ |
| $A_L$ | $\frac{1}{\sigma_Y}-\theta$ |
| $A_E$ | $\frac{\sigma_Y-1}{\sigma_Y}G_L+(1-\theta)G_E$ |
| $A_B$ | $\left(\frac{\sigma_Y-1}{\sigma_Y}G_L+(1-\theta)G_E\right)G_B+\frac{\sigma_E-1}{\sigma_E}G_S$ |
| $A_S, S$ | $\left(\frac{1}{\sigma_E}-\frac{1}{\sigma_Y}\right)G_L+\left(\frac{1}{\sigma_E}-\theta\right)G_E$ |

Table 2.1: Derivative signs

table 2.1 expressions with the same sign as (2.17) are collected. The calculations can be found in appendix 2.A.1.

The results under assumptions (2.13), (2.14) and (2.15) can be summarized in the following proposition.

**Proposition 2.2.** *Assume that* (2.13), (2.14) *and* (2.15) *hold and that* $B_t > 0$ *and* $B_T > 0$ *for* $t \neq T$. *Then*

$$\frac{dB_t}{dA_{L,T}} \leq 0 \text{ and } \frac{dB_T}{dA_{L,T}} \geq 0$$

*and*

$$\frac{dB_t}{dX_T} \geq 0 \text{ and } \frac{dB_T}{dX_T} \leq 0 \text{ for } X \in \{A_Y, A_E, A_S, S\},$$

*while the signs of the effects of changes in* $A_{B,T}$ *are ambiguous.*

*Proof.* Follows from proposition 2.1 and table 2.1. □

These results can be intuitively understood in terms of the two effects (described in (2.9)) of changing the marginal product of fossil fuel and changing the marginal utility of consumption. The way that the exogenous variables are defined, increasing them always increases production, and therefore also consumption. This decreases the marginal utility of consumption which has a negative effect on the marginal value of using fossil fuel. The strength of this effect is determined by $\theta$.

Increasing $A_Y$ increases the marginal product of fossil fuel. The relative strength of the decrease in marginal utility and the increase in the marginal product of fossil fuel is determined by $\theta$ and under assumption (2.14) the net effect is negative.

Increasing $A_L$ increases the marginal product of fossil fuel. The strength of this effect is determined by the degree of complementarity as measured by $\sigma_Y$. So the sign of the net effect depends on the relative strength of the effects. Under assumption (2.15) the net effect is positive.

Increasing $A_E$ generates a direct positive effect on the marginal productivity of fossil fuel but also a negative effect on the marginal value of energy, since the amount of energy increases relative to to the amount of complementary inputs, and consumption. Under assumptions (2.13) and (2.14) the negative effect will always dominate.

When increasing the supply of alternative energy $A_S S$, there will be a negative effect on the marginal value of consumption and the marginal productivity of energy (since it increases the amount of energy compared to the amount of complementary inputs). It does, however, also have a positive effect on the marginal product of fossil fuel in the production of the composite energy good. For the parameter assumptions here, the negative effects dominate.

The effect of improving the state of the fossil-fuel based technology $A_B$ is not unambiguously determined by the parameter assumptions made. Increasing $A_B$ has a direct effect of increasing the marginal product of fossil fuel. It also decreases the marginal product of the composite energy good and the marginal value of consumption. Which of these effects will dominate depends on the values of the variables. Under assumptions (2.13) and (2.14) the first term in the derivative will be negative while the second term will be positive. Some further insight can be gained by fixing $G_E$ and $G_L$, implying that $Y$ is also fixed. The relative importance of fossil-fuel based versus alternative energy can then be varied (that is vary $G_B$ and $G_S$ in such a way that $G_E$ remains fixed). From the expression it can be seen that if a large share of energy comes from fossil fuel ($G_B >> G_S$) the net effect will be negative. In this case, varying $A_B$ is similar to varying $A_E$. If, on the other hand, a large share of the energy comes from alternative-energy sources ($G_S >> G_B$), then the net effect will be positive. Assuming that alternative energy will become more important relative to fossil-fuel based energy over time, the effect of an increase in the future value of $A_B$ on the value of fossil-fuel use will tend to be negative (giving an increase in the short run fossil-fuel use) if the change occurs soon, while it will tend to be positive (giving a decrease in the short run fossil-fuel use) if the change occurs in the distant future.

In conclusion, the sign of the effect of changing the future state of technology depends on which specific technology changes. The Green Paradox says that investing in alternative-energy technology will increase fossil-fuel use in the short run, which is true also in this model. If investment in that technology crowds out investments in other types of technology, this crowding out can amplify or dampen the Green Paradox effect depending on which type of technology is crowded out. If labor intensive technology $A_L$ is crowded out, this will amplify the effect of the future increase in $A_S$. If, instead, $A_Y$ or $A_E$ are crowded out, this will dampen the effect of the change in $A_S$. If it crowds out investment in $A_B$, this can either dampen or amplify the effect of the future change in $A_S$. To understand the effect of crowding out research on $A_B$, consider a situation where increased spending on research on alternative-energy technology crowds out spending on fossil-fuel based technology. This will not affect the current state of technology. When the current research spending starts to have a significant effect on the state of technology, the world may still be in a situation where fossil fuels are the dominant energy source. In that case the worsening of the state of the fossil-fuel based technology may decrease the supply of energy enough so that fossil fuel will be reallocated from both the present and the distant future to the intermediate future. Whether or not this will occur is a quantitative issue.

Summing up, this section gives the basic results concerning the effect of the future state of technology on fossil-fuel use, as described in proposition 2.2. An improvement in the future state of TFP, energy-saving technology or alternative energy will increase fossil-fuel use in the short run. An improvement in the future state of the labor augmenting technology will decrease fossil-fuel use in the short run. The effect of an improvement in the state of technology for fossil-fuel based energy generation is ambiguous.

## 2.3 Two-period model with capital

So far, fossil-fuel use has been the only endogenous variable. In this section I will also include capital and investments will be endogenously determined. For the general functional form, this complicates the analysis significantly. In this section I will therefore use a two-period model where the endogenous choices are the division of fossil-fuel use between the first and second periods and how much, out of first-period production, to invest into second-period capital. In section 2.5 I will instead assume that $\theta = \sigma_Y = 1$ but use an infinite time horizon.

### 2.3.1 Model setup

Let production depend on fossil-fuel use $B$, capital $K$ and a vector of exogenously given variables $\Gamma$. Capital will be assumed to be combined with labor according to a Cobb-Douglas production function into an input that is complementary to energy. The exogenous variables in $\Gamma$ will be the same as above with the exception that the technology factor for the complementary input (previously $A_L$) will now be called $A_{KL}$, since it gives the productivity of the combination of capital and labor.

The production function is

$$F(B, K; \Gamma) = A_Y \left[ \gamma_{KL} \left( A_{KL} K^{\alpha} L^{1-\alpha} \right)^{\frac{\sigma_Y - 1}{\sigma_Y}} + \gamma_E \left( A_E Y_E \right)^{\frac{\sigma_Y - 1}{\sigma_Y}} \right]^{\frac{\sigma_Y}{\sigma_Y - 1}}, \tag{2.18}$$

where $Y_E$ is the same as before and defined in (2.12).

Total fossil-fuel supply is $Q$ and, since there are no externalities, all of it will be used. The initial capital stock, $K_1$, is given and there is full depreciation of capital between periods. This gives the following set of equations:

$$\begin{aligned} C_1 &= F(B_1, K_1; \Gamma_1) - K_2 \\ C_2 &= F(B_2, K_2; \Gamma_2) \\ B_2 &= Q - B_1. \end{aligned}$$

### 2.3.2 Equilibrium

Since there are no externalities in this model, the planner solution will still coincide with the competitive equilibrium. The planner problem is to maximize the discounted sum of utility from consumption in the two periods. Assuming that fossil-fuel use is strictly positive in both periods (a sufficient condition for this to hold is that $\sigma_E < \infty$) the planner problem is

$$\max_{B_1, K_2} U\left(F\left(B_1, K_1; \Gamma_1\right) - K_2\right) + \beta U\left(F\left(Q - B_1, K_2; \Gamma_2\right)\right)$$

The first-order conditions read

$$\begin{aligned} B_1 &: U_1' F_{B,1} = \beta U_2' F_{B,2} \\ K_2 &: U_1' = \beta U_2' F_{K,2}. \end{aligned}$$

These can be rewritten to give the optimality conditions

$$F_{B,2} = F_{K,2} F_{B,1} \tag{2.19}$$

$$U_1' = \beta U_2' F_{K,2}, \tag{2.20}$$

where the first condition is the Hotelling rule and the second condition is the Euler equation. The Hotelling rule says that, at the margin, one unit of fossil fuel should contribute equally to second-period production if it is used in production in period 2 or if it is used in production in period 1 and the resulting production is invested into second period capital.

### 2.3.3 Changes in $\Gamma_2$

For a given $\Gamma_2$, (2.19) and (2.20) can, in principle, be solved for the equilibrium values of $B_1$ and $K_2$. This means that the system implicitly defines $B_1$ and $K_2$ as functions of $\Gamma_2$. In order to see how the equilibrium values depend on $\Gamma_2$, consider a change in $X$ which is one of the variables in $\Gamma_2$. The change in $X$ will induce endogenous changes in $B_1$ and $K_2$. Let primes denote derivatives with respect to $X$.

Differentiating both sides of (2.19) with respect to $X$, and noting that $B_2' = -B_1'$, we obtain

$$\begin{aligned}
\frac{d}{dX}F_{B,2} &= F_{B,2}\left[\frac{F_{BK,2}}{F_{B,2}}K_2' + \frac{F_{BB,2}}{F_{B,2}}B_2' + \frac{F_{BX,2}}{F_{B,2}}\right] = \{B_2' = -B_1'\} = \\
&= F_{B,2}\left[\frac{F_{BK,2}}{F_{B,2}}K_2' - \frac{F_{BB,2}}{F_{B,2}}B_1' + \frac{F_{BX,2}}{F_{B,2}}\right] \\
\frac{d}{dX}F_{K,2}F_{B,1} &= F_{K,2}F_{B,1}\left[\frac{F_{KK,2}}{F_{K,2}}K_2' + \frac{F_{BK,2}}{F_{K,2}}B_2' + \frac{F_{KX,2}}{F_{K,2}} + \frac{F_{BB,1}}{F_{B,1}}B_1'\right] \\
&= F_{B,2}\left[\frac{F_{KK,2}}{F_{K,2}}K_2' + \left(\frac{F_{BB,1}}{F_{B,1}} - \frac{F_{BK,2}}{F_{K,2}}\right)B_1' + \frac{F_{KX,2}}{F_{K,2}}\right].
\end{aligned}$$

These derivatives must be equal; equating them and rewriting gives

$$K_2' = \frac{\frac{F_{BB,2}}{F_{B,2}} - \frac{F_{BK,2}}{F_{K,2}} + \frac{F_{BB,1}}{F_{B,1}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}}B_1' + \frac{\frac{F_{KX,2}}{F_{K,2}} - \frac{F_{BX,2}}{F_{B,2}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}}. \tag{2.21}$$

Similarly, both sides of condition (2.20) can be differentiated with respect to $X$

$$\begin{aligned}
\frac{d}{dX}U_1' &== U_1'\frac{U_1''}{U_1'}\left[F_{B,1}B_1' - K_2'\right] = U_1'\theta\frac{1}{C_1}\left[K_2' - F_{B,1}B_1'\right] \\
\frac{d}{dX}\beta U_2'F_{K,2} &= \beta U_2'F_{K,2}\frac{U_2''}{U_2'}\left(F_{B,2}B_2' + F_{K,2}K_2' + F_{X,2}\right) \\
&\quad + \beta U_2'F_{K,2}\left[\frac{F_{KK,2}}{F_{K,2}}K_2' + \frac{F_{BK,2}}{F_{K,2}}B_2' + \frac{F_{KX,2}}{F_{K,2}}\right] \\
&= U_1'\left[\left(\frac{F_{KK,2}}{F_{K,2}} - \theta\frac{F_{K,2}}{F_2}\right)K_2' - \left(\frac{F_{BK,2}}{F_{K,2}} - \theta\frac{F_{B,2}}{F_2}\right)B_1'\right] \\
&\quad + U_1'\left[\frac{F_{KX,2}}{F_{K,2}} - \theta\frac{F_{X,2}}{F_2}\right].
\end{aligned}$$

Again, these derivatives must be the same; equating them and rewriting gives

$$K_2' = \frac{\theta\frac{1}{C_1}F_{B,1} + \theta\frac{F_{B,2}}{F_2} - \frac{F_{BK,2}}{F_{K,2}}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}}B_1' + \frac{\frac{F_{KX,2}}{F_{K,2}} - \theta\frac{F_{X,2}}{F_2}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}}. \tag{2.22}$$

Equalizing the right-hand sides of (2.21) and (2.22) and rewriting gives

$$B_1' = \frac{\frac{\frac{F_{KX,2}}{F_{K,2}} - \frac{F_{BX,2}}{F_{B,2}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}} - \frac{\frac{F_{KX,2}}{F_{K,2}} - \theta\frac{F_{X,2}}{F_2}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}}}{\frac{\theta\frac{1}{C_1}F_{B,1} + \theta\frac{F_{B,2}}{F_2} - \frac{F_{BK,2}}{F_{K,2}}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}} - \frac{\frac{F_{BB,2}}{F_{B,2}} - \frac{F_{BK,2}}{F_{K,2}} + \frac{F_{BB,1}}{F_{B,1}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}}}.$$

Rearranging the numerator and using that $\frac{F_{BK}}{F_B} = \frac{1}{\sigma_Y}\frac{F_K}{F}$ gives

$$\begin{aligned} B_1' = & \frac{\left(\frac{F_{KK,2}}{F_{K,2}} - \theta\frac{F_{K,2}}{F_2}\right)\frac{F_{BX,2}}{F_{B,2}} - \left(\frac{1}{\sigma_Y} - \theta\right)\frac{F_{KX,2}}{F_2} + \theta\left(\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}\right)\frac{F_{X,2}}{F_2}}{\frac{\theta\frac{1}{C_1}F_{B,1} + \theta\frac{F_{B,2}}{F_2} - \frac{F_{BK,2}}{F_{K,2}}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}} + \frac{\frac{F_{BK,2}}{F_{K,2}} - \frac{F_{BB,2}}{F_{B,2}} - \frac{F_{BB,1}}{F_{B,1}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}}} \\ & + \frac{\theta\frac{1}{C_1}\left(\frac{F_{KX,2}}{F_{K,2}} - \frac{F_{BX,2}}{F_{B,2}}\right)}{\frac{\theta\frac{1}{C_1}F_{B,1} + \theta\frac{F_{B,2}}{F_2} - \frac{F_{BK,2}}{F_{K,2}}}{\theta\frac{1}{C_1} + \theta\frac{F_{K,2}}{F_2} - \frac{F_{KK,2}}{F_{K,2}}} + \frac{\frac{F_{BK,2}}{F_{K,2}} - \frac{F_{BB,2}}{F_{B,2}} - \frac{F_{BB,1}}{F_{B,1}}}{\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}}}. \end{aligned}$$

The denominator (which does not contain any derivatives with respect to $X$ and therefore is the same regardless of which variable $X$ represents) can be shown to be positive.[1]

This means that the sign of $B_1'$ depends on the sign of the two expressions

$$\frac{F_{KX,2}}{F_{K,2}} - \frac{F_{BX,2}}{F_{B,2}} \tag{2.23}$$

and

$$\begin{aligned} & \left(\frac{F_{KK,2}}{F_{K,2}} - \theta\frac{F_{K,2}}{F_2}\right)\frac{F_{BX,2}}{F_{B,2}} - \left(\frac{1}{\sigma_Y} - \theta\right)\frac{F_{KX,2}}{F_2} \\ & + \theta\left(\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}\right)\frac{F_{X,2}}{F_2}. \end{aligned} \tag{2.24}$$

If the expressions have the same sign, then $B_1'$ will also have that sign. Otherwise the sign will depend on the relative strength of the terms.

In table 2.2, expressions with the same signs as expression (2.23) and (2.24) respectively, when differentiating with respect to different $X$, can be found (see 2.A.2 for the calculations). In the table, $G_B$, $G_E$ and $G_S$ are defined as in (2.16) while $G_{KL}$ is

$$G_{KL} = \gamma_{KL}\left(A_{KL}K^{\alpha}L^{1-\alpha}\right)^{\frac{\sigma_Y - 1}{\sigma_Y}}. \tag{2.25}$$

[1] Adding up the two ratios, the denominator (in the denominator) is positive and so are all terms in the numerator except for $-\frac{F_{BK,2}}{F_{K,2}}\left(\frac{F_{BK,2}}{F_{B,2}} - \frac{F_{KK,2}}{F_{K,2}}\right)$. The term $\frac{F_{BK,2}}{F_{K,2}}\frac{F_{KK,2}}{F_{K,2}}$ is cancelled by an equal term of opposite sign. The other term can be cancelled by showing that $\frac{F_{BB,2}F_{KK,2}}{F_{B,2}F_{K,2}} - \frac{F_{BK,2}^2}{F_{B,2}F_{K,2}} > 0$.

| wrt | sign of (2.23) | sign of (2.24) |
|---|---|---|
| $A_Y$ | 0 | $\theta - 1$ |
| $A_{KL}$ | $\frac{\sigma_Y - 1}{\sigma_Y}$ | $\left(\theta - \frac{1}{\sigma_Y}\right)$ |
| $A_E$ | $\frac{1-\sigma_Y}{\sigma_Y}$ | $\frac{1-\sigma_Y}{\sigma_Y}\left(1+\alpha\left(\theta-1\right)\right)G_{KL} + \left(\theta-1\right)\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)G_E$ |
| $A_B$ | $\frac{1-\sigma_Y}{\sigma_Y}G_B + \frac{1-\sigma_E}{\sigma_E}G_S$ | $\left[\begin{pmatrix} (\theta-1)\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)G_E \\ +\frac{1-\sigma_Y}{\sigma_Y}\left(1+\alpha(\theta-1)\right)G_{KL}\end{pmatrix} G_B - \begin{pmatrix}\left(1+\alpha(\theta-1)\right)G_{KL} \\ +\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)G_E\end{pmatrix}\frac{\sigma_E-1}{\sigma_E}G_S\right]$ |
| $A_S, S$ | $\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}$ | $\left(\theta - \frac{1}{\sigma_E}\right)\left(1-\alpha\frac{\sigma_Y-1}{\sigma_Y}\right)G_E + \left(\frac{1}{\sigma_Y}-\frac{1}{\sigma_E}\right)\left(1+\alpha\left(\theta-1\right)\right)G_{KL}$ |

Table 2.2: Derivative signs

**Proposition 2.3.** *Assume that* (2.13)*,* (2.14) *and* (2.15) *hold. Then* $\frac{dB_1}{dA_{KL,2}} \leq 0$*,* $\frac{dB_1}{dX} \geq 0$ *for* $X \in \{A_{Y,2}A_{E,2}A_{S,2}, S_2\}$ *and the sign of* $\frac{dB_1}{dA_{B,2}}$ *is ambiguous.*

*Proof.* Follows from inspection of table 2.2. □

Thus, qualitatively, the results are the same here as in proposition 2.2. When looking at changes in $A_B$, the results are, as in section 2.2.4, ambiguous. As in that case, consider a situation where $G_{KL}$ and $G_E$ are fixed. The sign of the effect will depend on the relative sizes of $G_B$ and $G_S$. For both (2.23) and (2.24), the factor in front of $G_B$ is positive while the factor in front of $G_S$ is negative. This means that $B_1'$ will be positive if $G_B >> G_S$ and negative if $G_S >> G_B$. This is the same qualitative behavior as in section 2.2.4.

In conclusion, the results from section 2.2.4 hold also in this two-period model with capital.

## 2.4 Introducing climate change

I will now introduce climate change into the model. Climate change is an externality that comes from the amount of greenhouse gases building up in the atmosphere as a result of, among other things, the burning of fossil fuels. Introducing climate change related externalities into the model will allow me to look at a number of different issues. When there are externalities in the model, the decentralized equilibrium and the planner solution will no longer coincide. In a decentralized equilibrium, the externalities are not internalized in agents' decisions, but there will still be general equilibrium effects that will be taken into account. In the planner solution, the climate effects will be fully internalized. Furthermore, as long as all fossil fuel is used in the planner solution, it can be implemented as a decentralized equilibrium with taxation. In this section I will consider a model with infinite time horizon, but without capital. I will then look at how the allocation of fossil-fuel use depends on the sequence $\{\Gamma_t\}_{t=0}^{\infty}$ both in the decentralized equilibrium and in the planner solution. Furthermore, I will comment on how the optimal tax rate and the welfare gains from implementing the optimal tax scheme depend on $\{\Gamma_t\}_{t=0}^{\infty}$.

### 2.4.1 Modeling climate change

Climate change is driven by the concentration of greenhouse gases. Carbon dioxide, $CO_2$, that is emitted from the burning of fossil fuels is an important greenhouse gas. As the concentration of greenhouse gases increases in the atmosphere, the radiative balance between earth and the surrounding space changes. Primarily, the amount of radiation leaving the earth's atmosphere decreases. This leads to an increase in the temperature.

The full process of climate change is very complicated. Here I will assume that the climate state can be described as the concentration of greenhouse gases in the atmosphere. What primarily matters for the climate is the temperature in the atmosphere. For a given concentration of greenhouse gases, the atmospheric temperature will reach a steady state relatively quickly. Thus, if a time period is long enough, assuming that the atmospheric temperature is in steady state is not a bad assumption. This modeling abstracts the temperature changes in the oceans, which occur much more slowly.

Carbon that is emitted, in the form of $CO_2$, will partly be absorbed either in the biosphere, through photosynthesis, or into the oceans. The processes through which carbon moves between different reservoirs are collectively referred to as the carbon cycle. I will, for most of the derived results, assume that the carbon cycle is linear. This assumption is also made by Nordhaus in the DICE/RICE models. There are some reasons why this assumption may not be fulfilled. The oceans could become saturated, meaning that their capacity to take up carbon from the atmosphere decreases, and the increased temperature could alter the behavior of the biosphere so that it becomes a source, rather than a sink, of carbon in the future. These things aside, I do believe that the model of the climate that I use here serves as a useful approximation. It can, for instance, be shown that a model of the type used here can replicate the behavior of the climate system in the DICE/RICE models quite well (see Golosov et al., 2011).

I will first describe how I model the carbon cycle and then describe how I assume that the climate state translates into effects on the economy through the damage function.

**The carbon cycle**

Let $M_t$ be the climate state in period $t$: the stock of greenhouse gases in the atmosphere. I will assume that this stock, in period $t$, moves towards a steady-state value of $\bar{M}_t$. This steady-state value will depend on past emissions. The steady-state level could depend in many ways on the entire history of emissions. I will assume that it depends on the total amount of past emissions. This means that $\bar{M}_0$ is given and that for $t > 0$, $\bar{M}_t = \bar{M}\left(\sum_{t'=0}^{t-1} B_{t'}\right)$. The steady-state stock is assumed to be (weakly) increasing in the total amount of past emissions, that is $\bar{M}'\left(\sum_{t'=0}^{t-1} B_{t'}\right) \geq 0$. I will further assume that the movement towards the steady-state is linear. This gives the law of motion for the stock of greenhouse gases as

$$M_{t+1} = M_t + B_t - \delta\left[M_t - \bar{M}\left(\sum_{t'=0}^{t-1} B_{t'}\right)\right], \tag{2.26}$$

where $\delta$ is the rate at which carbon is removed from the atmosphere through the carbon cycle.[2]

Defining $\bar{M}_t = \bar{M}\left(\sum_{t'=0}^{t-1} B_{t'}\right)$ and taking $M_0$ as given, gives that the climate state in period $t+1$ is

$$M_{t+1} = (1-\delta)^{t+1} M_0 + \sum_{t'=0}^{t} (1-\delta)^{t-t'}\left(B_{t'} + \delta \bar{M}_{t'}\right).$$

Thus, fossil-fuel use in period $T$ affects climate from period $T+1$ and onwards. When considering the effects of changes in the pattern of fossil-fuel use, induced by changes in the exogenously given variables, the derivative of the climate state with respect to emissions is needed. This derivative is

$$\frac{dM_t}{dB_T} = \begin{cases} 0 & \text{if } t \leq T \\ (1-\delta)^{t-1-T} + \delta \sum_{u=T}^{t-1} (1-\delta)^{t-1-u} \frac{d\bar{M}_u}{dB_T} & \text{if } t > T \end{cases},$$

the latter of which is positive.

[2] Here, the climate state evolves according to a law of motion for that state variable. An alternative description would be to let the climate state depend on the history of fossil-fuel use. What matters for most of the results below is that the effect of fossil-fuel use in period $t_1$ on the climate state in period $t_2 > t_1$ only depends on $t_2 - t_1$.

A further assumption that I will make for some of the analysis is that $\bar{M}$ is linear

$$\bar{M}\left(\sum_{t'=0}^{t-1} B_{t'}\right) = \bar{M}_0 + m\sum_{t'=0}^{t-1} B_{t'}. \tag{2.27}$$

This corresponds to a fraction $m$ of emissions staying in the atmosphere forever. With this shape of $\bar{M}$, for $t > 0$ we have

$$\begin{aligned} M_t &= (1-\delta)^t M_0 + \sum_{t'=0}^{t-1}\left((1-\delta)^{t-1-t'} + \delta m \sum_{u=0}^{t-2-t'} (1-\delta)^u\right) B_{t'} \\ &\quad + \delta \sum_{t'=0}^{t-1} (1-\delta)^{t'} \bar{M}_0 \\ &= \sum_{t'=0}^{t-1}\left(m + (1-m)(1-\delta)^{t-1-t'}\right) B_{t'} \\ &\quad + \bar{M}_0 + (1-\delta)^t\left(M_0 - \bar{M}_0\right). \end{aligned} \tag{2.28}$$

Two relatively similar special cases are $\delta = 0$ and $m = 1$. Setting $\delta = 0$ gives

$$M_t = M_0 + \sum_{t'=0}^{t-1} B_{t'}. \tag{2.29}$$

When $\delta = 0$, no carbon is taken up in the carbon cycle. Therefore emissions stay in the atmosphere forever and all emissions add to the climate state.

Setting $m = 1$ instead, we obtain

$$M_t = \bar{M}_0 + (1-\delta)^t (M_0 - \bar{M}_0) + \sum_{t'=0}^{t-1} B_{t'}. \tag{2.30}$$

Here, all emissions still stay forever in the atmosphere since they add to the equilibrium concentration. In addition, any difference between actual and equilibrium concentration in period 0 decays over time. If $M_0 = \bar{M}_0$ the two cases are equivalent.

Recent research (see Archer, 2005) indicates that a significant share of current emissions will stay in the atmosphere for a significant time, so making one of these assumptions may not be a bad approximation.

Under the linearity assumption of $\bar{M}$, the derivative of the climate state with respect to past fossil-fuel use is

$$\frac{dM_t}{dB_T} = \begin{cases} 0 & \text{if } t \leq T \\ m + (1-m)(1-\delta)^{t-1-T} & \text{if } t > T \end{cases}. \tag{2.31}$$

For both of the cases $\delta = 0$ and $m = 1$, $\frac{dM_t}{dB_T} = 1$ for $t > T$.

Thus, if $\bar{M}$ is assumed to be linear and given by (2.27), the climate state in period $t$ is given by (2.28) and the effect of fossil-fuel use in period $T$ on the climate state in period $t$ is given by (2.31).

If, furthermore, either

$$\delta = 0 \text{ or } m = 1 \text{ and } M_0 \leq \bar{M}_0, \tag{2.32}$$

then

$$t_2 \geq t_1 \Rightarrow M_{t_2} \geq M_{t_1} \tag{2.33}$$

and

$$\frac{dM_t}{dB_T} = \begin{cases} 0 \text{ if } t \leq T \\ 1 \text{ if } t > T \end{cases}. \tag{2.34}$$

### The damage function

The climate can affect the economy in a number of different ways. Here, I will, as in the DICE/RICE-models and in Golosov et al. (2011), assume that climate affects the economy as a multiplicative factor on production. That is, climate change affects total factor productivity.

Consumption is equal to production:

$$C_t = Y_t = D(M_t)F(B_t; \Gamma_t), \tag{2.35}$$

where $D(M)$ is the damage function. I will assume that $D(M) \in (0, 1]$ and that $D'(M) < 0$. When it comes to the second derivative, it is not obvious what the sign should be. In the DICE and RICE models, damages are a concave function of the temperature, while temperature is a convex (logarithmic) function of the concentration of greenhouse gases in the atmosphere.

I will often use the assumption that

$$\frac{d}{dM}\frac{D'(M)}{D(M)} \leq 0. \tag{2.36}$$

Examples of damage functions that fulfill this assumption includes all concave functions ($D''(M) \leq 0$) and $D(M) = e^{-\kappa M}$.

Under this assumption.

$$\frac{d}{dM}\frac{D'(M)}{D(M)} = \frac{D''(M)}{D(M)} - \frac{D'(M)^2}{D(M)^2} = \frac{D'(M)}{D(M)}\left(\frac{D''(M)}{D'(M)} - \frac{D'(M)}{D(M)}\right) \leq 0$$
$$\Rightarrow \quad \frac{D''(M)}{D'(M)} - \frac{D'(M)}{D(M)} \geq 0$$

Thus, if the damage function fulfills (2.36) and $\theta \geq 1$ then

$$M_{t_2} \geq M_{t_1} \Rightarrow \frac{D'(M_{t_1})}{D(M_{t_1})} - \frac{D'(M_{t_2})}{D(M_{t_2})} \geq 0 \tag{2.37}$$

and

$$\frac{D''(M)}{D'(M)} - \frac{D'(M)}{D(M)} \geq 0 \text{ and } \frac{D''(M)}{D'(M)} - \theta\frac{D'(M)}{D(M)} \geq 0. \tag{2.38}$$

For some derivations I will make the more specific assumption

$$D(M) = e^{-\kappa M}; \tag{2.39}$$

with this damage function

$$\frac{D'(M)}{D(M)} = \frac{D''(M)}{D'(M)} = -\kappa \tag{2.40}$$

This completes the description of how climate change enters the model.

As emphasized above, when climate change is included as an externality in the model, the planner solution and the competitive equilibrium will no longer coincide and they need to be solved for separately. This is the subject of the following sections.

### 2.4.2 Decentralized equilibrium

I will now specify and solve for a competitive equilibrium. I will consider an equilibrium with a tax on sales of fossil fuel. For most of this section, I will treat taxes as given. In section 2.4.4 I will then look at the taxes that implement the optimal solution.

To begin with, I will specify who receives what income and who makes which decisions. I will assume that households derive income from supplying labor and from profits in the energy supplying firms (given a constant returns to scale production technology, the final good producing firms will not make any profits). Furthermore, the government's tax revenues are paid back as lump-sum amounts to the households. Let $g_t$ be the lump-sum transfer in period $t$.

Without any assumed disutility from work, labor will be inelastically supplied in quantity $L$. The energy supplying firms are assumed to be owned in equal shares by all households. The households can also save in the form of a riskless bond. Let $\pi_{B,t}$ and $\pi_{S,t}$ be the profits from fossil-fuel and alternative-energy supplying firms, respectively. Let $\frac{1}{r_{t+1}}$ be the price in period $t$ of a riskless bond that pays one unit of consumption in period $t+1$ and let $a_t$ be the holding of riskless bonds that pays in period $t$.

The budget constraint of the households is

$$C_t + \frac{1}{r_{t+1}} a_{t+1} = \pi_{B,t} + \pi_{S,t} + w_t + l_t + g_t + a_t. \tag{2.41}$$

A competitive equilibrium with taxation consists of sequences of quantities $\{B_t, C_t\}_{t=0}^{\infty}$, fossil-fuel taxes $\{\tau_t\}_{t=0}^{\infty}$, lump-sum tax rebates $\{g_t\}_{t=0}^{\infty}$ and prices $\{p_{B,t}, p_{S,t}, r_{t+1}, w_t\}_{t=0}^{\infty}$ such that

- households maximize their total utility $\sum_{t=0}^{\infty} \beta^t U(C_t)$ subject to the budget constraint (2.41)
- the bond market clears: $a_t = 0$ for all $t$
- prices are competitively determined
- fossil-fuel extracting firms maximize the discounted sum of profits from extraction
- the government balances its budget in each period

**Solving for the decentralized equilibrium**

**Households** The utility maximization problem of the representative household is

$$\max_{\{C_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$
$$\text{s.t. } C_t + \frac{1}{r_{t+1}} a_{t+1} = w_t L + a_t + \pi_{B,t} + \pi_{S,t} + g_t \, \forall t.$$

The Lagrangian of the problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \left( \beta^t U(C_t) + \lambda_t \left[ w_t L + a_t + \pi_{B,t} + \pi_{S,t} + g_t - C_t - \frac{1}{r_{t+1}} a_{t+1} \right] \right).$$

The first-order conditions read

$$C_t : \beta^t U'(C_t) = \lambda_t$$
$$a_{t+1} : \lambda_t \frac{1}{r_{t+1}} = \lambda_{t+1}.$$

Combining these conditions determines the price of the riskless bond:

$$r_{t+1} = \frac{1}{\beta} \frac{U'(C_t)}{U'(C_{t+1})}. \tag{2.42}$$

Market clearing in the bond market ($a_t = 0$) gives that

$$C_t = w_t L_t + \pi_{B,t} + \pi_{S,t} + g_t. \tag{2.43}$$

**Competitive prices** Assuming perfect competition, the prices of the inputs are given by their respective marginal products

$$p_{B,t} = D(M_t)F_{B,t},\ p_{S,t} = D(M_t)F_{S,t} \text{ and } w_t = D(M_t)F_{L,t}. \tag{2.44}$$

**Balanced government budget** The government tax revenues, that are paid back as lump sums to the households, are

$$g_t = \tau_t p_{B,t} B_t. \tag{2.45}$$

**Fossil-fuel extracting firms** The fossil-fuel suppliers are assumed to maximize discounted profits from extraction over time. The discount rate used between profits in periods $t$ and $t+1$ is $r_{t+1}$. Since there are no extraction costs, the profit that the firm makes, per extracted unit of fossil fuel, in period $t$ is $(1-\tau_t)p_{B,t}$. The maximization problem of a fossil-fuel extracting firm, that has a fossil-fuel resource with quantity $q$, is

$$\max_{\{b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[\prod_{t'=0}^{t-1} \frac{1}{r_{t'+1}}\right] (1-\tau_t)p_{B,t}b_t \text{ s.t. } \sum_{t=0}^{\infty} b_t \leq q.$$

The Lagrangian of this problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\prod_{t'=0}^{t-1} \frac{1}{r_{t'+1}}\right] p_{B,t}b_t + \lambda \left[q - \sum_{t=0}^{\infty} b_t\right] + \mu_t b_t.$$

The first-order condition with respect to $b_t$ reads

$$\left[\prod_{t'=0}^{t-1} \frac{1}{r_{t'+1}}\right] (1-\tau_t)p_{B,t} = \lambda - \mu_t. \tag{2.46}$$

Using (2.42), we obtain

$$\prod_{\tau=0}^{t-1} \frac{1}{r_{\tau+1}} = \beta\frac{U'(C_1)}{U'(C_0)}\beta\frac{U'(C_2)}{U'(C_1)} \cdots \beta\frac{U'(C_t)}{U'(C_{t-1})} = \beta^t\frac{U'(C_t)}{U'(C_0)}.$$

For any $t$ and $T$ such that $b_t > 0$ and $b_T > 0$, this implies that

$$\beta^t\frac{U'(C_t)}{U'(C_0)}(1-\tau_t)p_{B,t} = \beta^T\frac{U'(C_T)}{U'(C_0)}(1-\tau_T)p_{B,T}.$$

or

$$\beta^t U'(C_t)(1-\tau_t)p_{B,t} = \beta^T U'(C_T)(1-\tau_T)p_{B,T}. \tag{2.47}$$

At the aggregate level, this condition must hold between any periods $t$ and $T$ such that $B_t > 0$ and $B_T > 0$.

**Equilibrium conditions** Without any extraction costs for fossil fuel, or the alternative-energy inputs, the profits that the households receive in period $t$ are

$$\pi_{B,t} = (1-\tau_t)p_{B,t}B_t$$
$$\pi_{S,t} = p_{S,t}S_t$$

Substituting these expressions, and the lump-sum transfers from (2.45), into the households' budget constraint (2.43) and using the fact that the production function has constant returns to scale in $L$, $B$, and $S$, we see that

$$\begin{aligned} C_t &= D(M_t)F_{L,t}L_t + (1-\tau_t)D(M_t)F_{B,t}B_t \\ &\quad + D(M_t)F_{S,t}S_t + \tau_t D(M_t)F_{B,t}B_t \\ &= D(M_t)\left(F_{L,t}L_t + F_{B,t}B_t + F_{S,t}S_t\right) \\ &= D(M_t)F(B_t;\Gamma_t) = Y_t. \end{aligned}$$

Thus, consumption is equal to production. Substituting this quantity into the Hotelling rule (2.47) gives

$$\beta^t U'(Y_t)(1-\tau_t)p_{B,t} = \beta^T U'(Y_T)(1-\tau_T)p_{B,T}.$$

Using the definition of $H$ from (2.5), the Hotelling rule can be written

$$\beta^t(1-\tau_t)H_t = \beta^T(1-\tau_T)H_T, \tag{2.48}$$

where $H_t = H(B_t;\Gamma_t)$.

All fossil fuel will always be extracted in a decentralized equilibrium with competitive fossil-fuel supply since, as long as the fossil-fuel price is strictly positive, increased extraction in any period increases profits.

In sum, the decentralized equilibrium is characterized by the Hotelling rule, (2.48) and $\sum_{t=0}^{\infty} B_t = Q_0$.

So far the equilibrium looks exactly like the equilibrium without externalities. This is because the decisions do not internalize the climate effects. However, the factor $D(M_t)$ is endogenously determined and will affect the equilibrium fossil-fuel use. This will become apparent in the next section.

### Changes in $\{\Gamma_t\}_{t=0}^{\infty}$

I will now consider changes in $X_T$, which is one of the variables in $\Gamma_T$, for some $T > 0$ such that $B_T > 0$. This change will induce endogenous changes in $\{B_t\}_{t=0}^{\infty}$. When $X_T$ changes, the equilibrium conditions will still be satisfied. Thus, when differentiating with respect to $X_T$, the derivative of both sides of the Hotelling rule (2.48) must must be the same for all $t$ such that $B_t > 0$.

Let the full derivatives of $H_t$, $B_t$, $M_t$ and $C_t$ with respect to $X_T$ be denoted by primes. I will show that, in some specified cases, $B'_t$ will have the same sign for all $t < T$ and that if $B'_T$ does not have the opposite sign, then $B'_t$ will have the same sign for all $t$ which would violate the constraint on the total amount of available fossil fuel. The difference, compared to section 2.2.4, is that changes in fossil-fuel use in one period will change the damages, and therefore potentially the value of fossil-fuel use, in all future periods.

I will start by differentiating the Hotelling rule to get a relationship between changes in $B_t$ in different periods. I will then show what these relationships lead to in two special cases.

With $H_t$ given by (2.5), the derivative of $H_t$ with respect to $X_T$ is

$$H'_t = \begin{cases} H_t \left[\frac{U''_t}{U'_t} C'_t + \frac{D'_t}{D_t} M'_t + \frac{F_{BB,t}}{F_{B,t}} B'_t\right] & \text{if } t \neq T \\ H_T \left[\frac{U''_T}{U'_T} C'_T + \frac{D'_T}{D_T} M'_T + \frac{F_{BB,T}}{F_{B,T}} B'_T + \frac{F_{BX,T}}{F_{B,T}}\right] & \text{if } t = T \end{cases},$$

where the derivative of consumption is

$$C'_t = \frac{d}{dX_t} D(M_t) F_t = \begin{cases} C_t \left[\frac{D'_t}{D_t} M'_t + \frac{F_{B,t}}{F_t} B'_t\right] & \text{if } t \neq T \\ C_T \left[\frac{D'_T}{D_T} M'_T + \frac{F_{B,T}}{F_T} B'_T + \frac{F_{X,T}}{F_T}\right] & \text{if } t = T \end{cases}.$$

Using the form of the utility function (2.10), we obtain

$$\frac{U''}{U'} C = -\theta.$$

Also, to simplify the notation, define

$$\xi_{B,t} = \theta \frac{F_{B,t}}{F_t} - \frac{F_{BB,t}}{F_{B,t}} > 0 \text{ and } \xi_{X,t} = \frac{F_{BX,t}}{F_{B,t}} - \theta \frac{F_{X,t}}{F_t}. \tag{2.49}$$

Note that $\xi_X$ has the same sign as (2.17), the expression that determines the sign of the effects in the model without externality in proposition 2.2.

Using this notation, substituting $C'_t$ in $H'_t$ and rewriting gives

$$H'_t = \begin{cases} H_t \left[(1-\theta) \frac{D'_t}{D_t} M'_t - \xi_{B,t} B'_t\right] & \text{if } t \neq T \\ H_T \left[(1-\theta) \frac{D'_T}{D_T} M'_T - \xi_{B,T} B'_T + \xi_{X,T}\right] & \text{if } t = T \end{cases}.$$

The first two terms relate to the effects of the endogenous changes in $\{B_t\}$. The first term captures the change in the externality. If $M_t$ increases, this decreases production in period $t$ and decreases the marginal product of fossil fuel. Which of these effects dominates, and consequently whether the marginal value of fossil-fuel use increases or decreases, depends on the value of $\theta$. In the second term, the factor in front of $B_t'$ is negative, capturing that the value of fossil-fuel use decreases as the use increases. The third term in the expression for $t = T$ captures the direct effect of the change in $X_T$ on $H_T$. The effect consists of an effect on the marginal product of fossil fuel and an effect on production that affects the marginal value of consumption.

The effects of the change in $X_T$ can be summarized in the conditions

$$\beta^{t_1}(1-\tau_{t_1})H_{t_1}' = \beta^{t_2}(1-\tau_{t_2})H_{t_2}' \tag{2.50}$$

for all $t_1$ and $t_2$ such that $B_{t_1} > 0$ and $B_{t_2} > 0$ and

$$\sum_{t=0}^{\infty} B_t' = 0.$$

From the Hotelling rule (2.48), $\beta^{t_1}(1-\tau_{t_1})H_{t_1} = \beta^{t_2}(1-\tau_{t_2})H_{t_2}$. The comparison between the changes in the two time periods will depend on whether one of them is $T$ or not.

Consider first the case where $t_1 \neq T$ and $t_2 \neq T$. Then

$$(1-\theta)\frac{D_{t_1}'}{D_{t_1}}M_{t_1}' - \xi_{B,t_1}B_{t_1}' = (1-\theta)\frac{D_{t_2}'}{D_{t_2}}M_{t_2}' - \xi_{B,t_2}B_{t_2}'.$$

Solving for $B_{t_2}'$ gives

$$B_{t_2}' = \frac{\xi_{B,t_1}}{\xi_{B,t_2}}B_{t_1}' + (1-\theta)\frac{\frac{D_{t_2}'}{D_{t_2}}M_{t_2}' - \frac{D_{t_1}'}{D_{t_1}}M_{t_1}'}{\xi_{B,t_2}}. \tag{2.51}$$

Consider now the case $t_1 = t$ and $t_2 = T$. Then

$$(1-\theta)\frac{D_t'}{D_t}M_t' - \xi_{B,t}B_t' = (1-\theta)\frac{D_T'}{D_T}M_T' - \xi_{B,T}B_T' + \xi_{X,T}.$$

Solving for $B_T'$ gives

$$B_T' = \frac{\xi_{B,t}}{\xi_{B,T}}B_t' + (1-\theta)\frac{\frac{D_T'}{D_T}M_T' - \frac{D_t'}{D_t}M_t'}{\xi_{B,T}} + \frac{\xi_{X,T}}{\xi_{B,T}}. \tag{2.52}$$

These conditions can now be used to see how the pattern of fossil-fuel use changes in response to the change in $X_T$. We will first look at logarithmic utility and then consider higher curvature.

### The log-utility case

Assume first that utility is logarithmic: $\theta = 1$. The second terms in (2.51) and (2.52) will then be zero. If considering the case $t_1 \neq T$ and $t_2 \neq T$, then (2.51) implies that

$$B'_{t_2} = \frac{\xi_{B,t_1}}{\xi_{B,t_2}} B'_{t_1}. \tag{2.53}$$

Thus, $B'_{t_1}$ and $B'_{t_2}$ must have the same sign. This leads to the following proposition

**Proposition 2.4.** *If $\theta = 1$, then a change in $X_T$ will have the following effect on fossil-fuel use*

$$\text{Sgn}\left(B'_t\right) = \text{Sgn}\left(\frac{F_{X,T}}{F_T} - \frac{F_{BX,T}}{F_{B,T}}\right) \text{ for any } t \neq T \text{ such that } B_t > 0$$

*and*

$$\text{Sgn}\left(B'_T\right) = \text{Sgn}\left(\frac{F_{BX,T}}{F_{B,T}} - \frac{F_{X,T}}{F_T}\right).$$

*Proof.* From (2.53) it follows that $B'_t$ must have the same sign for all $t \neq T$. Since $\sum_{t=0}^{\infty} B'_t = 0$, $B'_T$ must have the opposite sign. From (2.52), with $\theta = 1$, it follows that $B'_T$ can only have the opposite sign to $B'_t$ if $B'_T$ has the same sign as $\xi_{X,T}$. The proposition then follows from the definition of $\xi_{X,T}$, (2.49). □

Note that this proposition has the same implications as proposition 2.2. This is because when utility is assumed to be logarithmic, the externalities disappear from the decentralized equilibrium and therefore the effects are the same as in the model without externalities. This independence of externalities can be interpreted in two different ways. The first interpretation is that the effects on the marginal productivity of fossil fuel and the marginal utility of consumption exactly cancel each other out when utility is logarithmic. The second interpretation is that if utility is logarithmic, then the externality enters as an additive term in the utility function. A different way to make the damages irrelevant for the decentralized equilibrium is to assume that damages enters as an additive term in the utility function instead of affecting productivity. If this assumption is made, the externalities will not affect the decentralized equilibrium and consequently the effect will be the same as in the model without externalities regardless of the value of $\theta$.

### The case $\theta > 1$ and $\delta = 0$ or $m = 1$ and $M_0 \leq \bar{M}_0$

When $\theta \neq 1$, the damages do not cancel from conditions (2.51) and (2.52) since the effects on marginal product of fossil fuel and on marginal utility of consumption no longer cancel. If $\theta > 1$, the effect on marginal utility of consumption dominates the effect on marginal productivity of fossil fuel and higher damages increase the value of fossil-fuel use. The opposite holds if $\theta < 1$. When the effects on the climate state must be taken into account, the analysis of the effects of a change in $X_T$ on fossil-fuel use becomes, in general, significantly more complicated since the effects of changes in fossil-fuel use in some period on the damages in all future periods must be taken into account. In this section, I will consider the case $\theta > 1$ and show that if the damage function fulfills (2.36) and the carbon cycle fulfills (2.32), that is, either $\delta = 0$ or $m = 1$ and $M_0 \leq \bar{M}_0$, then the sign of $B'_t$ will be the same for all $t < T$ and $B'_T$ must have the opposite sign.

Consider a change in $X_T$ and a time period $t_2 > 0$ such that $t_2 \neq T$, $B_{t_2} > 0$ and such that $B'_t$ has the same sign for all $t < t_2$. For this case (2.51) gives that

$$B'_{t_2} = \frac{\xi_{B,t_1}}{\xi_{B,t_2}} B'_{t_1} + (\theta - 1) \frac{\frac{D'_{t_1}}{D_{t_1}} M'_{t_1} - \frac{D'_{t_2}}{D_{t_2}} M'_{t_2}}{\xi_{B,t_2}} \tag{2.54}$$

for some $t_1 < t_2$ such that $B_{t_1} > 0$ and $t_1 \neq T$. The first term, as before, has the same sign as $B'_{t_1}$. The second term captures the change in relative profitability of using fossil fuel in the two periods that comes from the changes in damages caused by the change in fossil-fuel use. When $\theta > 1$, the second term has the same sign as

$$\frac{D'_{t_1}}{D_{t_1}} M'_{t_1} - \frac{D'_{t_2}}{D_{t_2}} M'_{t_2}.$$

Under assumption (2.32)

$$M'_t = \sum_{t'=0}^{t-1} B'_{t'}$$

giving that

$$M'_{t_2} = M'_{t_1} + \sum_{t=t_1}^{t_2-1} B'_t$$

implying that

$$\frac{D'_{t_1}}{D_{t_1}} M'_{t_1} - \frac{D'_{t_2}}{D_{t_2}} M'_{t_2} = \frac{D'_{t_1}}{D_{t_1}} M'_{t_1} - \frac{D'_{t_2}}{D_{t_2}} \left( M'_{t_1} + \sum_{t=t_1}^{t_2-1} B'_t \right).$$

Substituting this into (2.54) delivers

$$B'_{t_2} = \frac{\xi_{B_{t_1}}}{\xi_{B_{t_2}}} B'_{t_1} + \frac{\theta - 1}{\xi_{B,t_2}} \left[ \left( \frac{D'_{t_1}}{D_{t_1}} - \frac{D'_{t_2}}{D_{t_2}} \right) M'_{t_1} - \frac{D'_{t_2}}{D_{t_2}} \sum_{t=t_1}^{t_2-1} B'_t \right].$$

Under assumption (2.32), $M_{t_2} \geq M_{t_1}$. Since the damage function is assumed to fulfill (2.36), (2.37) implies that the factor in front of $M'_{t_1}$ is positive. By assumption, $B'_{t_1}$, $M'_{t_1}$ and $B'_t$ all have the same sign. Since the factors in front of each of them are positive, $B'_{t_2}$ must also have that same sign.

This leads to the following proposition:

**Proposition 2.5.** *If $\theta > 1$, the damage function fulfills* (2.36) *and the carbon cycle fulfills* (2.32)*, then the effects of a change in $X_T$ will be such that*

$$\operatorname{Sgn}(B'_t) = \operatorname{Sgn}\left(\left(\theta \frac{F_{X,T}}{F_T} - \frac{F_{BX,T}}{F_{B,T}}\right)\right) \text{ for any } t < T \text{ such that } B_t > 0$$

*and*

$$\operatorname{Sgn}(B'_T) = \operatorname{Sgn}\left(\left(\frac{F_{BX,T}}{F_{B,T}} - \theta \frac{F_{X,T}}{F_T}\right)\right)$$

*Proof.* From the argument above, it follows that $B'_t$ will have the same sign for all $t < T$. If $B_T$ had the same sign then $B'_t$ would have the same sign for all $t$ which would violate the condition that $\sum_{t=0}^{\infty} B'_t = 0$. This means that $B'_T$ must have the opposite sign compared to $B'_t$ for any $t < T$. From (2.52) it follows that this change in sign must come from $\xi_{X,T}$ and therefore that $B'_T$ must have the same sign as $\xi_{X,T}$. The proposition then follows from the definition of $\xi_{X,T}$ in (2.49). □

Compared to the case without externalities, it is more difficult to predict the effects for $t > T$ since then the changes in damages enter the comparisons.

In conclusion, proposition 2.5 again confirms the results of section 2.2.4.

By closer inspection of the signs of $B'_t$ for $t \leq T$, it is clear that proposition 2.5 should hold under much more general conditions. The details of this argument, however, is not outlined here. What is required is that a sufficient amount of $CO_2$ stays in the atmosphere for a sufficient amount of time.

### 2.4.3 Planner solution

The section above dealt with effects of introducing externalities in the decentralized equilibrium. Since the climate effects are not internalized directly in any decisions made, the effects on the equilibrium allocation came through indirect effects. In the planner solution, the effects of fossil-fuel use on the climate are taken into account directly. One implication of this is that it is no longer certain that the planner will choose to use all the fossil fuel. For much of the analysis I will, however, focus on the case where the constraint on total fossil-fuel supply does bind.

#### Characterizing the planner solution

The planner wants to choose fossil-fuel use to maximize the discounted sum of utility for the representative household in the economy. Consumption is, in each period, given by (2.35) and the law of motion of the climate state is (2.26). Furthermore, the planner is constrained by the total supply of fossil fuel. As in the decentralized equilibrium, I will still assume that $\{\Gamma_t\}_{t=0}^{\infty}$ is an exogenously given sequence. This gives the following formulation of the planner problem:

$$\begin{aligned}
\max_{\{B_t\}_{t=0}^{\infty}} \ & \sum_{t=0}^{\infty} \beta^t U(C_t) \\
\text{s.t} \ & C_t = D(M_t) F(B_t; \Gamma_t) \\
& \sum_{t=0}^{\infty} B_t \leq Q_0 \\
& M_{t+1} = M_t + B_t - \delta \left[ M_t - \bar{M} \left( \sum_{t'=0}^{t-1} B_{t'} \right) \right] \\
& B_t \geq 0.
\end{aligned}$$

The Lagrangian of this problem can be written

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U\left(D(M_t) F(B_t; \Gamma_t)\right) + \lambda \left[ Q_0 - \sum_{t=0}^{\infty} B_t \right] + \mu_t B_t,$$

where $\lambda$ is positive if the constraint on total available fossil fuel binds and $\mu_t$ is positive if the non-negativity constraint on fossil-fuel use binds in period $t$. While not written out explicitly here, the law of motion of the climate state makes $M_t$ a function of $B_0, B_1, \ldots, B_{t-1}$.

Taking the first order condition with respect to $B_t$ gives

$$\lambda - \mu_t = \beta^t U'_t D_t F_{B,t} + \sum_{t'=t+1}^{\infty} \beta^{t'} U'_{t'} F_{t'} D'_{t'} \frac{dM_{t'}}{dB_t}.$$

The first term on the right hand side is the marginal discounted utility of the marginal product of fossil-fuel use in period $t$. The sum is the total discounted marginal utility effects, in all future periods, caused by emissions in period $t$.

To simplify notation, let

$$H_{1,t} = U_t' D_t F_{B,t} \text{ and } H_{2,t,t'} = \beta^{t'} U_{t'}' F_{t'} D_{t'}' \frac{dM_{t'}}{dB_t}. \tag{2.55}$$

Here $H_{1,t}$ is the same as $H_t$, defined in (2.5).

The only difference between $H_{2,t,t'}$ for different values of $t$ comes from the factor $\frac{dM_{t'}}{dB_t}$. If $\bar{M}$ is assumed to be linear and given by (2.27), then $\frac{dM_{t'}}{dB_t}$ is given by (2.31) and is therefore a constant that depends only on the difference between $t$ and $t'$.

Using (2.55), the optimality condition can be written

$$\beta^t H_{1,t} + \sum_{t'=t+1}^{\infty} H_{2,t,t'} = \lambda - \mu_t. \tag{2.56}$$

When the constraint on total amount of available fossil fuel does not bind, $\lambda = 0$ and the solution is characterized by

$$\beta^t H_{1,t} + \sum_{t'=t+1}^{\infty} H_{2,t,t'} = 0 \tag{2.57}$$

for all $t$ such that $B_t > 0$.

When the constraint binds, the solution is characterized by (2.56) and $\sum_{t=0}^{\infty} B_t = Q_0$. Consider $t_1 < t_2$ such that $B_{t_1} > 0$ and $B_{t_2} > 0$. Then

$$\beta^{t_1} H_{1,t_1} + \sum_{t'=t_1+1}^{\infty} H_{2,t_1,t'} = \beta^{t_2} H_{1,t_2} + \sum_{t'=t_2+1}^{\infty} H_{2,t_2,t'}. \tag{2.58}$$

This equation says that for any two periods where fossil-fuel use is positive, the marginal value of fossil-fuel use, net of future damages caused by emissions, should be the same.

### Changes in $\{\Gamma_t\}_{t=0}^{\infty}$

Consider again a change in $X_T$ which is one of the exogenous variables in $\Gamma_T$. This change will induce changes in the sequence of fossil-fuel use $\{B_t\}_{t=0}^{\infty}$. Letting primes denote total derivatives with respect to $X_T$ (taking the response of fossil-fuel use into account), the terms in the optimality conditions (2.55) will be affected as follows:

$$H_{1,t}' = \begin{cases} H_{1,t} \left[ \frac{U_t''}{U_t'} C_t' + \frac{D_t'}{D_t} M_t' + \frac{F_{BB,t}}{F_{B,t}} B_t' \right] & \text{if } t \neq T \\ H_{1,T} \left[ \frac{U_T''}{U_T'} C_T' + \frac{D_T'}{D_T} M_T' + \frac{F_{BB,T}}{F_{B,T}} B_T' + \frac{F_{BX,T}}{F_{B,T}} \right] & \text{if } t = T \end{cases}$$

and

$$H'_{2,t,t'} = \begin{cases} H_{2,t,t'} \left[ \frac{U''_{t'}}{U'_{t'}} C'_{t'} + \frac{F_{B,t'}}{F_{t'}} B'_{t'} + \frac{D''_{t'}}{D'_{t'}} M'_{t'} \right] & \text{if } t' \neq T \\ H_{2,t,T} \left[ \frac{U''_T}{U'_T} C'_T + \frac{F_{B,T}}{F_T} B'_T + \frac{D''_T}{D'_T} M'_T + \frac{F_{X,T}}{F_T} \right] & \text{if } t' = T \end{cases}.$$

As in the decentralized equilibrium, the change in consumption is

$$C'_t = \begin{cases} C_t \left[ \frac{F_{B,t}}{F_t} B'_t + \frac{D'_t}{D_t} M'_t \right] & \text{if } t \neq T \\ C_T \left[ \frac{F_{B,T}}{F_T} B'_T + \frac{D'_T}{D_T} M'_T + \frac{F_{X,T}}{F_T} \right] & \text{if } t = T \end{cases}.$$

The shape of the utility function (2.10) implies that $\frac{U''_t}{U'_t} C_t = -\theta$. Using this, and the notation defined in (2.49), the changes in $H_{1,t}$ and $H_{2,t,t'}$ can be rewritten as

$$H'_{1,t} = \begin{cases} H_{1,t} \left[ (1-\theta) \frac{D'_t}{D_t} M'_t - \xi_{B,t} B'_t \right] & \text{if } t \neq T \\ H_{1,T} \left[ (1-\theta) \frac{D'_T}{D_T} M'_T - \xi_{B,T} B'_T + \xi_{X,T} \right] & \text{if } t = T \end{cases} \tag{2.59}$$

and

$$H'_{2,t,t'} = \begin{cases} H_{2,t,t'} \left[ (1-\theta) \frac{F_{B,t'}}{F_{t'}} B'_{t'} + \left( \frac{D''_{t'}}{D'_{t'}} - \theta \frac{D'_{t'}}{D_{t'}} \right) M'_{t'} \right] & \text{if } t' \neq T \\ H_{2,t,T} \left[ (1-\theta) \left( \frac{F_{B,T}}{F_T} B'_T + \frac{F_{X,T}}{F_T} \right) + \left( \frac{D''_T}{D'_T} - \theta \frac{D'_T}{D_T} \right) M'_T \right] & \text{if } t' = T \end{cases}. \tag{2.60}$$

The effect of a change in $X_T$ depends on whether the constraint on the total amount of available fossil fuel binds or not. If the constraint does not bind, then (2.57) gives that the change in the planner solution's allocation of fossil-fuel use in response to the change in $X_T$ can be described by

$$\beta^t H'_{1,t} + \sum_{t'=t+1}^{\infty} H'_{2,t,t'} = 0$$

for any $t$ such that $B_t > 0$. Note that, since the emissions in period $t$ will affect the climate state in all future periods, there can still be nontrivial interactions between the changes of $B_t$ in different periods.

Assume now that the constraint on the total amount of available fossil fuel binds. Using (2.58), the effects of changing $X_T$ can then be described by the equations

$$\beta^{t_1} H'_{1,t_1} + \sum_{t=t_1+1}^{\infty} H'_{2,t_1,t} = \beta^{t_2} H'_{1,t_2} + \sum_{t=t_2+1}^{\infty} H'_{2,t_2,t}$$

for any $t_1 < t_2$ such that $B_{t_1} > 0$ and $B_{t_2} > 0$ and

$$\sum_{t=0}^{\infty} B'_t = 0.$$

The term

$$\frac{H'_{2,t,t'}}{H_{2,t,t'}} = \begin{cases} (1-\theta)\frac{F_{B,t'}}{F_{t'}} B'_{t'} + \left(\frac{D''_{t'}}{D'_{t'}} - \theta \frac{D'_{t'}}{D_{t'}}\right) M'_{t'} & \text{if } t' \neq T \\ (1-\theta)\left(\frac{F_{B,T}}{F_T} B'_T + \frac{F_{X,T}}{F_T}\right) + \left(\frac{D''_T}{D'_T} - \theta \frac{D'_T}{D_T}\right) M'_T & \text{if } t' = T \end{cases} \tag{2.61}$$

depends on $t'$ but is independent of $t$. The difference between $H'_{2,t_1,t'}$ and $H'_{2,t_2,t'}$ can then be written

$$\begin{aligned} H'_{2,t_1,t'} - H'_{2,t_2,t'} &= H_{2,t_1,t'} \frac{H'_{2,t_1,t'}}{H_{2,t_1,t'}} - H_{2,t_2,t'} \frac{H'_{2,t_2,t'}}{H_{2,t_2,t'}} \\ &= \left(H_{2,t_1,t'} - H_{2,t_2,t'}\right) \frac{H'_{2,t_1,t'}}{H_{2,t_1,t'}} \\ &= \beta^{t'} U'_{t'} F_{t'} D'_{t'} \left(\frac{dM_{t'}}{dB_{t_1}} - \frac{dM_{t'}}{dB_{t_2}}\right) \frac{H'_{2,t_1,t'}}{H_{2,t_1,t'}}. \end{aligned}$$

It follows that

$$\begin{aligned} \beta^{t_2} H'_{1,t_2} - \beta^{t_1} H'_{1,t_1} &= \sum_{t=t_1+1}^{t_2} H'_{2,t_1,t} + \sum_{t=t_2+1}^{\infty} \left(H'_{2,t_1,t} - H'_{2,t_2,t}\right) \\ &= \sum_{t=t_1+1}^{t_2} H'_{2,t_1,t} + \sum_{t=t_2+1}^{\infty} \left(H_{2,t_1,t} - H_{2,t_2,t}\right) \frac{H'_{2,t_1,t}}{H_{2,t_1,t}} \\ &= \sum_{t=t_1+1}^{t_2} H'_{2,t_1,t} \\ &\quad + \sum_{t=t_2+1}^{\infty} \beta^t U'_t F_t D'_t \frac{H'_{2,t_1,t}}{H_{2,t_1,t}} \left(\frac{dM_t}{dB_{t_1}} - \frac{dM_t}{dB_{t_2}}\right) \end{aligned} \tag{2.62}$$

where $\frac{H'_{2,t_1,t'}}{H_{2,t_1,t'}}$ is given by (2.61).

I will now show what these conditions imply in some special cases.

**The log-utility case**

Assume first that utility is logarithmic: $\theta = 1$.

Then (2.55) becomes

$$H_{1,t} = \frac{F_{B,t}}{F_t} > 0 \text{ and } H_{2,t,t'} = \beta^{t'} \frac{D'_{t'}}{D_{t'}} \frac{dM_{t'}}{dB_t} < 0. \tag{2.63}$$

Equation (2.59) gives

$$H'_{1,t} = \begin{cases} -H_{1,t}\xi_{B,t}B'_t & \text{if } t \neq T \\ H_{1,T}\left[-\xi_{B,T}B'_T + \xi_{X,T}\right] & \text{if } t = T \end{cases} \tag{2.64}$$

while equation (2.60) delivers

$$H'_{2,t,t'} = H_{2,t,t'}\left(\frac{D''_{t'}}{D'_{t'}} - \frac{D'_{t'}}{D_{t'}}\right)M'_{t'}. \tag{2.65}$$

Condition (2.62) then becomes

$$\begin{aligned} \beta^{t_2}H_{1,t_2}\xi_{B,t_2}B'_{t_2} = {} & \beta^{t_1}H_{1,t_1}\xi_{B,t_1}B'_{t_1} \\ & - \sum_{t=t_2+1}^{\infty}\beta^t\frac{D'_t}{D_t}\left(\frac{D''_t}{D'_t} - \frac{D'_t}{D_t}\right)M'_t\left(\frac{dM_t}{dB_{t_1}} - \frac{dM_t}{dB_{t_2}}\right) \\ & - \sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t} - \frac{D'_t}{D_t}\right)M'_t \end{aligned} \tag{2.66}$$

if $t_1 \neq T$ and $t_2 \neq T$. If, instead $t_1 = t$ and $t_2 = T$, (2.62) becomes

$$\begin{aligned} \beta^T H_{1,T}\xi_{B,T}B'_T = {} & \beta^t H_{1,t}\xi_{B,t}B'_t + \beta^T H_{1,T}\xi_{X,T} \\ & - \sum_{t'=T+1}^{\infty}\beta^{t'}\frac{D'_{t'}}{D_{t'}}\left(\frac{D''_{t'}}{D'_{t'}} - \frac{D'_{t'}}{D_{t'}}\right)M'_{t'}\left(\frac{dM_{t'}}{dB_t} - \frac{dM_{t'}}{dB_T}\right) \\ & - \sum_{t'=t+1}^{T}H_{2,t,t'}\left(\frac{D''_{t'}}{D'_{t'}} - \frac{D'_{t'}}{D_{t'}}\right)M'_{t'}. \end{aligned} \tag{2.67}$$

In general, this condition depends on the entire future, since emissions made in any period has effects over the entire future.

If the carbon cycle fulfills (2.32), (2.34) then implies that all terms of the first sum are zero. If instead the damage function is assumed to be exponential, as in (2.39), (2.40) implies that all terms of both sums are zero.

If the carbon cycle fulfills (2.32), (2.66) becomes

$$\beta^{t_2}H_{1,t_2}\xi_{B,t_2}B'_{t_2} = \beta^{t_1}H_{1,t_1}\xi_{B,t_1}B'_{t_1} + \sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D'_t}{D_t} - \frac{D''_t}{D'_t}\right)M'_t.$$

Assume further that the damage function fulfills (2.36) and that $B'_t$ has the same sign for all $t < t_2$. Combining $H_{2,t_1,t} \leq 0$ with (2.38) gives that all the factors in front of $M'_{t'}$ in the sum will be positive. The assumption that $B'_t$ has the same sign for all $t < t_2$ implies that $M'_t$ will also have that same sign for all $t \leq t_2$, since $M'_t = \sum_{t'=0}^{t-1} B'_{t'}$. So the conclusion is that $B'_{t_2}$ will have that sign as well.

If, instead, the damage function is exponential, given by (2.39), (2.66) becomes

$$H_{1,t_2}\xi_{B,t_2}B'_{t_2} = H_{1,t_1}\xi_{B,t_2}B'_{t_1} \tag{2.68}$$

and $B'_t$ will have the same sign for all $t \neq T$.

This gives the following proposition:

**Proposition 2.6.** *Assume that utility is logarithmic, $\theta = 1$, and that the damage function fulfills* (2.36)*. Assume further at least one of $\delta = 0$, $m = 1$ or that the damage function is exponential, given by* (2.39)*. Consider a change in $X_T$, which is one of the variables in $\Gamma_T$. Then the induced changes in fossil-fuel use are such that*

$$\mathrm{Sgn}\left(B'_t\right) = \mathrm{Sgn}\left(\frac{F_{X,T}}{F_T} - \frac{F_{BX,T}}{F_{B,T}}\right) \text{ for any } t < T \text{ such that } B_t > 0$$

*and*

$$\mathrm{Sgn}\left(B'_T\right) = \mathrm{Sgn}\left(\frac{F_{BX,T}}{F_{B,T}} - \frac{F_{X,T}}{F_T}\right)$$

*If the damage function is assumed to be exponential, given by* (2.39)*, the sign of $B'_t$ will be the same for all $t \neq T$.*

*Proof.* From the argument above, it follows that $B'_t$ must have the same sign for all $t \neq T$ in the exponential case. Since $\sum_{t=0}^{\infty} B'_t = 0$, $B'_T$ must have the opposite sign. In the case $\delta = 0$ or $m = 1$, $B'_t$ will have the same sign for all $t < T$. If $B_T$ also had that same sign, $B'_t$ would have the same sign for all $t$ contradicting $\sum_{t=0}^{\infty} B'_t = 0$. From (2.67), with the assumption $\delta = 0$ or $m = 1$

$$\begin{aligned}\beta^T H_{1,T}\xi_{B,T}B'_T = {} & \beta^t H_{1,t}\xi_{B,T}B'_t + \beta^T H_{1,T}\xi_{X,T} \\ & - \sum_{t'=t+1}^{T} H_{2,t,t'}\left(\frac{D''_{t'}}{D'_{t'}} - \frac{D'_{t'}}{D_{t'}}\right)M'_{t'}\end{aligned}$$

On the RHS the first and last terms both have the same sign as $B'_t$ for all $t < T$. It follows that $B'_T$ can only have the opposite sign to that of $B_t$ if

$$\mathrm{Sgn}\left(B'_t\right) = \mathrm{Sgn}\left(-\xi_{X,T}\right) = \mathrm{Sgn}\left(\frac{F_{X,T}}{F_T} - \frac{F_{BX,T}}{F_{B,T}}\right)$$

□

In conclusion, the conclusions from section 2.2.4 hold here as well.

**The case $\theta > 1$ and either $\delta = 0$ or $m = 1$ and $\bar{M}_0 \geq M_0$**

I will now show that for $\theta > 1$, and if the carbon cycle fulfills (2.32), the changes in $B'_t$, for all $t < T$, induced by a change in $\Gamma_T$, must have the same sign.

Consider $t_1 \neq T$ such that $B_{t_1} > 0$ and $t_2 \neq T$ such that $t_2$ is the smallest $t > t_1$ with $B_t > 0$. This implies that for any $t$ such that $t_1 < t \leq t_2$, $M_t \geq M_{t_1}$ and $M'_t = M'_{t_1} + B'_{t_1}$. I also assume that $B'_t$ has the same sign for all $t \leq t_1$.

Also, under assumption (2.36), for $t > t_1$, (2.37) gives

$$M_t \geq M_{t_1} \Rightarrow \frac{D'_t}{D_t} \leq \frac{D'_{t_1}}{D_{t_1}}.$$

Using (2.34), (2.62) gives

$$\beta^{t_2} H'_{1,t_2} - \beta^{t_1} H'_{1,t_1} = \sum_{t=t_1+1}^{t_2} H'_{2,t_1,t}, \tag{2.69}$$

where

$$\begin{aligned}
H'_{1,t_1} &= H_{1,t_1} \left[ (1-\theta) \frac{D'_{t_1}}{D_{t_1}} M'_{t_1} - \xi_{B,t_1} B'_{t_1} \right] \\
H'_{1,t_2} &= H_{1,t_2} \left[ (1-\theta) \frac{D'_{t_2}}{D_{t_2}} M'_{t_2} - \xi_{B,t_2} B'_{t_2} \right] \\
&= H_{1,t_2} \left[ (1-\theta) \frac{D'_{t_2}}{D_{t_2}} \left( M'_{t_1} + B'_{t_1} \right) - \xi_{B,t_2} B'_{t_2} \right] \\
H'_{2,t_1,t} &= H_{2,t_1,t} \left[ (1-\theta) \frac{F_{B,t}}{F_t} B'_t + \left( \frac{D''_t}{D'_t} - \theta \frac{D'_t}{D_t} \right) M'_t \right].
\end{aligned}$$

Under the assumptions made here, $B'_t = 0$ if $t_1 < t < t_2$, and $M'_t = M'_{t_1} + B'_{t_1}$, implying

$$\begin{aligned}
\sum_{t=t_1+1}^{t_2} H'_{2,t_1,t} &= \sum_{t=t_1+1}^{t_2} H_{2,t_1,t} \left[ (1-\theta) \frac{F_{B,t}}{F_t} B'_t + \left( \frac{D''_t}{D'_t} - \theta \frac{D'_t}{D_t} \right) M'_t \right] \\
&= H_{2,t_1,t_2} (1-\theta) \frac{F_{B,t_2}}{F_{t_2}} B'_{t_2} \\
&\quad + \sum_{t=t_1+1}^{t_2} H_{2,t_1,t} \left( \frac{D''_t}{D'_t} - \theta \frac{D'_t}{D_t} \right) \left( M'_{t_1} + B'_{t_1} \right)
\end{aligned} \tag{2.70}$$

Substituting (2.70) into (2.69) and rewriting delivers (see appendix 2.A.3)

$$B'_{t_2} = \frac{\beta^{t_2} H_{1,t_2}(1-\theta)\frac{D'_{t_2}}{D_{t_2}} + \beta^{t_1} H_{1,t_1}\xi_{B,t_1} - \sum_{t=t_1+1}^{t_2} H_{2,t_1,t}\left(\frac{D''_t}{D'_t} - \theta\frac{D'_t}{D_t}\right)}{\beta^{t_2} H_{1,t_2}\xi_{B,t_2} + H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}} B'_{t_1}$$
$$+ \frac{\beta^{t_2} H_{1,t_2}(\theta-1)\left(\frac{D'_{t_1}}{D_{t_1}} - \frac{D'_{t_2}}{D_{t_2}}\right)}{\beta^{t_2} H_{1,t_2}\xi_{B,t_2} + H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}} M'_{t_1} \tag{2.71}$$
$$- \frac{\sum_{t=t_1+1}^{t_2} H_{2,t_1,t}\left(\frac{D''_t}{D'_t} - \frac{D'_t}{D_t} + (\theta-1)\left(\frac{D'_{t_1}}{D_{t_1}} - \frac{D'_t}{D_t}\right)\right)}{\beta^{t_2} H_{1,t_2}\xi_{B,t_2} + H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}} M'_{t_1}$$

Under the assumptions made here, the denominator and all the terms in the numerators are positive. To see this, first note that $D_t > 0$, $D'_t < 0$, $H_{1,t} > 0$, $H_{2,t_1,t} < 0$, $\xi_{B,t} > 0$ and $\theta > 1$; the result then follows from (2.33), (2.37) and (2.38). By assumption, $M'_{t_1}$ and $B'_{t_1}$ have the same sign. Since all the factors multiplying these factors in (2.71) are positive, $B'_{t_2}$ will also have that same sign.

This leads to the following proposition:

**Proposition 2.7.** *Assume that $\theta > 1$ and that the damage function fulfills (2.36). Assume further that $\delta = 0$ or that $m = 1$ and $\bar{M}_0 = M_0$. Consider a change in $X_T$ which is one of the variables in $\Gamma_T$. Then $B'_t$ will have the same sign for all $t < T$.*

Furthermore, consider a situation where $\Gamma_T$ changes in such a way that the value of using fossil fuel in period $T$ decreases. Could that lead to an increase in $B_t$ for $t < T$? I will argue here that this is unlikely. If fossil-fuel use was unaltered, the value of fossil-fuel use in all other periods would be higher than in period $T$. Assume then that $B_0$ decreased. Proposition 2.7 then implies that $B_t$ would decrease for all $t < T$. Since the climate state from period $T$ and onwards would then be improved, this should, if anything, decrease the value of fossil-fuel use also for $t > T$.

## 2.4.4 Optimal taxation

We have seen above that changes in the technology trends will change the pattern of fossil-fuel use over time, both in the socially optimal planner solution and in a decentralized equilibrium. This also has implications for how taxation should be used to move the competitive equilibrium towards the planner solution. In this section I will describe the implications for the optimal tax on fossil-fuel use.

In a model without any costs of extracting fossil fuel, the planner solution can be implemented in a competitive equilibrium with taxation if and only if all fossil fuel is exhausted in the planner solution.[3] So assume in the following that all fuels are exhausted in the planner solution.

Furthermore, in a model without extraction costs and without any distributional considerations, the optimal tax system is not uniquely determined. This is because the tax system need only affect the relative profitability of extraction in different periods. Therefore, only the relative tax rate across time periods, and not the level of the tax rate, is determined. Here I will look at the particular tax system that exactly internalizes the future climate related damages (along the equilibrium path) in the fossil-fuel price in period $t$. This tax system is robust to introducing extraction costs. Choosing different levels of taxes (but still maintaining the right intertemporal relation between the tax rate in different periods) would redistribute income to or from the fossil-fuel resource owners. In this model, the fossil-fuel resources are owned in equal shares by all the households and therefore these distributional concerns do not matter here.

Writing out the equilibrium condition (2.48), with $H_t$ given by (2.5), gives

$$\beta^{t_1} U'_{t_1} D_{t_1} F_{B,t_1} \left(1 - \tau_{t_1}\right) = \beta^{t_2} U'_{t_2} D_{t_2} F_{B,t_2} \left(1 - \tau_{t_2}\right). \tag{2.72}$$

Since this condition only depends on the ratio between $1-\tau_1$ and $1-\tau_2$, the same equilibrium allocation can be supported by any sequence of taxes that gives the same ratio for any $t_1$ and $t_2$; this is the indeterminacy described above.

The optimality condition from the planner solution (2.58) can be written

$$\beta^{t_1} U'_{t_1} D_{t_1} F_{B,t_1} + \sum_{t=t_1}^{\infty} \beta^t U'_t F_t D'_t \frac{dM_t}{dB_{t_1}} = \beta^{t_2} U'_{t_2} D_{t_2} F_{B,t_2} + \sum_{t=t_2}^{\infty} \beta^t U'_t F_t D'_t \frac{dM_t}{dB_{t_2}}.$$

The tax system that exactly internalizes future climate damages is such that, for the sequence $\{B_t\}_{t=0}^{\infty}$ that solves the planner problem, and any $t$ such that $B_t > 0$,

$$\beta^t U'_t D_t F_{B,t} (1 - \tau_t) = \beta^t U'_t D_t F_{B,t} + \sum_{t'=t}^{\infty} \beta^{t'} U'_{t'} F_{t'} D'_{t'} \frac{dM_{t'}}{dB_t}.$$

[3]If a proportional tax on fossil-fuel use was 100% in all time periods, then any allocation of fossil-fuel use can be an equilibrium; if the tax rate is lower than that in any period, then all fuels must be exhausted in the competitive equilibrium.

Simplified, this becomes

$$D_t F_{B,t} \tau_t = -\sum_{t'=t}^{\infty} \frac{\beta^{t'} U'_{t'}}{\beta^t U'_t} F_{t'} D'_{t'} \frac{dM_{t'}}{dB_t}. \tag{2.73}$$

This equation shows that the tax should balance the current value of using fossil fuel against the discounted sum of future damages.

The left-hand side of (2.73) is

$$D_t F_{B,t} \tau_t = \{(2.44)\} = p_{B,t} \tau_t \equiv \hat{\tau}_t.$$

This is the per unit tax in period $t$, in terms of period $t$ consumption (while $\tau_t$ is the tax rate in period $t$). The right-hand side of (2.73) gives the discounted sum of the value of future marginal damages caused by period $t$ emissions. These are expressed in terms of the value (in utility terms) of lost consumption from future damages. The damages are then normalized back to period $t$ consumption through division by the marginal utility of consumption in period $t$.

Consider now, again, a change in $X_T$ which is one of the variables in $\Gamma_T$. If this change decreases the value of using fossil fuel in period $T$, this should increase the relative value of using fossil fuel in another period $t$. This should lead to increased fossil-fuel use in that period which leads to decreased value of using fossil fuel in that period (both through decreased marginal productivity and decreased marginal utility from consumption). Assumption (2.36) implies that more emissions increases the relative marginal damages. Both of these effects suggest that increased fossil-fuel use in a period, in the planner solution, implies that the optimal tax in that period increases.

Under some specific assumptions it can be shown that this intuition holds. Assume that utility is logarithmic ($\theta = 1$) and that the damage function is exponential, as in (2.39). The optimal tax condition then becomes

$$\hat{\tau}_t = \frac{1}{\beta^t} D_t F_t \kappa \sum_{t'=t}^{\infty} \beta^{t'} \frac{dM_{t'}}{dB_t}.$$

If $\bar{M}$ is assumed to be linear, the sum in the RHS is a constant that depends only on $m$, $\delta$ and $\beta$. So the per unit tax depends on $D_t$ and $F_t$. If fossil-fuel use is increased in period $t$, this will tend to increase the tax. The damage $D_t$ can change in either way depending on all changes in the sequence $\{B_{t'}\}_{t'=0}^{t-1}$. For $t = 0$, $D_t$ will be unchanged.

Looking instead at the tax rate $\tau_t$, we have

$$\tau_t = \frac{1}{\beta^t} \frac{F_t}{F_{B,t}} \kappa \sum_{t'=t}^{\infty} \beta^{t'} \frac{dM_{t'}}{dB_t}$$

An increase in $B_t$ increases $F_t$ and decreases $F_{B,t}$; both changes go in the direction of increasing the tax rate.

The results of proposition 2.6 give the effect on fossil-fuel use of changes in $X_T$. This leads to the following proposition.

**Proposition 2.8.** *Assume that $\theta = 1$, $D(M) = e^{-\kappa M}$, that $\bar{M}$ is linear and that $\sum_{t=0}^{\infty} B_t \leq Q_0$ binds in the planner solution. Consider changes in $X_T$, where $B_T > 0$. Then, for any $t \neq T$ such that $B_t > 0$, the changes in the optimal tax rate $\tau$ and per unit tax $\hat{\tau}$, are such that*

$$\text{Sgn}\left(\hat{\tau}_0'\right) = \text{Sgn}\left(\tau_t'\right) = \text{Sgn}\left(\left(\frac{F_{X,T}}{F_T} - \frac{F_{BX,T}}{F_{B,T}}\right)\right)$$

*Proof.* Since, $\hat{\tau}_0$ and $\tau_t$ in periods $t \neq T$, the tax rate changes in the same direction as fossil-fuel use, the proposition follows from proposition 2.6. □

In section 2.5 I will show that this result generalizes to a model with the same assumptions regarding the utility function and damage function with capital as long as capital enters the production function as in a Cobb-Douglas production function and it depreciates fully.

### 2.4.5 Welfare effects of changing $\{\Gamma_t\}_{t=0}^{\infty}$

Changes in the exogenous variables $\{\Gamma_t\}_{t=0}^{\infty}$ will also have implications for welfare, both in the socially optimal planner solution and in the decentralized equilibrium. This will also have implications for the welfare gains from imposing the proper tax on fossil-fuel use in the decentralized equilibrium. An increase in any variable in $\Gamma$ in any time period will always have a direct positive effect by increasing production in that period. In the decentralized equilibrium there will also be indirect effects going through the changes in the fossil-fuel use pattern. Since the externalities are internalized in the planner solution, the envelope condition implies that there are no such indirect effects in the planner solution. In this section I will compare the planner solution to the unregulated decentralized equilibrium and see how changes in $\{\Gamma_t\}_{t=0}^{\infty}$ affect the welfare gains from going to the optimal planner solution from the unregulated decentralized equilibrium. Using a simplified example, I will show that the change in the welfare gains from taxation can have either sign.

Let $\{B_t^P\}_{t=0}^{\infty}$ and $\{B_t^D\}_{t=0}^{\infty}$ be the fossil-fuel allocation in the planner solution and the decentralized equilibrium respectively. Let also $\{C_t^P\}_{t=0}^{\infty}$ and $\{C_t^D\}_{t=0}^{\infty}$ be the corresponding consumption sequences. Welfare is then given by $V^P$ and $V^D$ respectively where

$$V^P = \sum_{t=0}^{\infty} \beta^t U(C_t^P) \text{ and } V^D = \sum_{t=0}^{\infty} \beta^t U(C_t^D).$$

Letting primes denote derivatives with respect to $X_T$, the welfare effects of a change in $X_T$ are then

$$\frac{dV}{dX_T} = \sum_{t=0}^{\infty} \beta^t U_t' C_t' = \sum_{t=0}^{\infty} \beta^t U_t' C_t \left[ \frac{F_{B,t}}{F_t} B_t' + \frac{D_t'}{D_t} M_t' \right] + \beta^T U_T' C_T \frac{F_{X,T}}{F_T}$$
$$= \sum_{t=0}^{\infty} \beta^t U_t' \left[ D_t F_{B,t} B_t' + F_t D_t' M_t' \right] + \beta^T U_T' D_T F_{X,T}. \tag{2.74}$$

The sum represents the indirect effects coming from redistribution of fossil-fuel use over time. The second part is the direct effect of the change in $X_T$.

In the planner solution, the indirect effects in (2.74) can be rewritten as

$$\sum_{t=0}^{\infty} \beta^t U_t' \left[ D_t F_{B,t} B_t' + F_t D_t' M_t' \right]$$
$$= \sum_{t=0}^{\infty} \beta^t U_t' \left[ D_t F_{B,t} B_t' + F_t D_t' \sum_{t'=0}^{t} \frac{dM_t}{dB_{t'}} B_{t'}' \right]$$
$$= \sum_{t=0}^{\infty} \beta^t U_t' D_t F_{B,t} B_t' + \sum_{t'=0}^{\infty} \sum_{t=t'}^{\infty} \beta^t U_t' F_t D_t' \frac{dM_t}{dB_{t'}} B_{t'}'$$
$$= \sum_{t=0}^{\infty} \beta^t U_t' D_t F_{B,t} B_t' + \sum_{t=0}^{\infty} \sum_{t'=t+1}^{\infty} \beta^{t'} U_{t'}' F_{t'} D_{t'}' \frac{dM_{t'}}{dB_t} B_t'$$
$$= \sum_{t=0}^{\infty} \left[ \beta^t U_t' D_t F_{B,t} + \sum_{t'=t+1}^{\infty} \beta^{t'} U_{t'}' F_{t'} D_{t'}' \frac{dM_{t'}}{dB_t} \right] B_t' = \{(2.55) \text{ and } (2.56)\}$$
$$\sum_{t=0}^{\infty} \left[ \lambda - \mu_t \right] B_t' = \sum_{t=0}^{\infty} \lambda B_t' - \sum_{t=0}^{\infty} \mu_t B_t' = \lambda \sum_{t=0}^{\infty} B_t' = 0.$$

Thus, as expected from the envelope theorem, the indirect effect is zero in the planner solution. This leaves only the direct effect

$$\frac{dV^P}{dX_T} = \beta^T U_T' D_T F_{X,T}. \tag{2.75}$$

In the decentralized equilibrium, using (2.5) and (2.48) with $\tau_t = 0$ for all $t$, the indirect effect in (2.74) can be rewritten as

$$\sum_{t=0}^{\infty} \beta^t U_t' \left[ D_t F_{B,t} B_t' + F_t D_t' M_t' \right] = \sum_{t=0}^{\infty} \beta^t H_t B_t' + \sum_{t=0}^{\infty} \beta^t U_t' F_t D_t' M_t'.$$

For any $t$ such that $B'_t \neq 0$, $\beta^t H_t$ will be the same. Assuming that $B_{\bar{t}} > 0$, $\beta^t H_t = \beta^{\bar{t}} H_{\bar{t}}$ for all $t$ such that $B'_t \neq 0$. This delivers

$$\sum_{t=0}^{\infty} \beta^t H_t B'_t = \beta^{\bar{t}} H_{\bar{t}} \sum_{t=0}^{\infty} B'_t = 0.$$

The implication of this finding is that the effect of the redistribution of fossil-fuel use in the decentralized equilibrium, in response to the change in $X_T$, is zero when considering the effects on the marginal product of fossil fuel. This is because the effect on the marginal value of fossil-fuel use in production is taken into account in the decentralized equilibrium. However, the effect on the externality is not taken into account and that effect will typically not be zero.

This gives the welfare effect of a change in $X_T$ in the decentralized equilibrium as

$$\frac{dV^D}{dX_T} = \sum_{t=0}^{\infty} \beta^t U'_t D'_t F_t M'_t + \beta^T U'_T D_T F_{X,T} \tag{2.76}$$

Since, by assumption, $F_{X,T} \geq 0$, the derivative of welfare with respect to $X_T$ will be positive in the planner solution. The same thing is not generally true in the decentralized equilibrium since there the climate effects are not taken into account and therefore the climate effects, and the total welfare change, could go either way.

Comparing the welfare effects of changes in the two solutions, the direct effect is present in both of them. If the solutions are not too different (e.g., if the externality is not too severe), the direct welfare effect will be similar in both solutions. If utility is logarithmic, what matters is how different $\frac{F_{X,T}}{F_T}$ is.

If the direct effect is similar in the decentralized equilibrium and the planner solution, the sign of the change in the difference between the welfare in the solutions, that is, the sign of the change in the welfare gains from taxation, will depend only on the sign of the indirect welfare effect going through the change in the externalities. Typically, a different measure of welfare comparisons, e.g., equivalent variation in consumption, is used instead of looking at the change in total welfare. However, under the functional-form assumptions made here the results from both procedures would be qualitatively the same.[4]

I will now show, by looking at a simple example, that the indirect effect can go either way. The simplifying assumptions I will make are $\theta = 1$, $\sigma_Y = 1$, $\sigma_E \to \infty$ and $D(M) = e^{-\kappa M}$. I will also assume that either $m = 1$ or $\delta = 0$.

[4] This is true since $\sum_t \beta^t \ln\left((1+\Delta_C)C_t\right) = \frac{\ln(1+\Delta_C)}{1-\beta} + \sum_t \beta^t \ln(C_t)$.

When the energy sources are perfect substitutes ($\sigma_E \to \infty$), fossil-fuel use will typically only be non-zero in a finite number of periods. I will assume that the solution is such that $B_t > 0$ if and only if $t \leq \bar{t}$.

I will then look at the indirect effect of a change in $X_T$, which is a variable in $\Gamma_T$ for some $T \leq \bar{t}$, on the welfare in the decentralized equilibrium. Under the assumptions made, the externality part of the welfare change in the decentralized equilibrium (2.76) becomes

$$\sum_{t=0}^{\infty} \beta^t U_t' D_t' F_t M_t' = \sum_{t=0}^{\infty} \beta^t \frac{D_t'}{D_t} M_t' = -\kappa \sum_{t=0}^{\infty} \beta^t M_t'.$$

Furthermore, under these assumptions, the production function can be written

$$F = A_Y L^{1-\gamma_E} (A_B B + A_S S)^{\gamma_E}$$

implying that

$$F_B = \gamma_E \frac{A_B F}{A_B B + A_S S} \text{ and } F_{BB} = (\gamma_E - 1) \frac{A_B F_B}{A_B B + A_S S}$$

and that

$$\frac{F_B}{F} = \gamma_E \frac{A_B}{A_B B + A_S S} \text{ and } \xi_B = \frac{F_B}{F} - \frac{F_{BB}}{F_B} = \frac{A_B}{A_B B + A_S S} = \frac{1}{\gamma_E} \frac{F_B}{F}.$$

For any $t_1 \leq \bar{t}$ and $t_2 \leq \bar{t}$, the equilibrium condition (2.48) becomes

$$\beta^{t_1} U'(C_{t_1}) D(M_{t_1}) F_{B,t_1} = \beta^{t_2} U'(C_{t_2}) D(M_{t_2}) F_{B,t_2}$$
$$\beta^{t_1} \frac{F_{B,t_1}}{F_{t_1}} = \beta^{t_2} \frac{F_{B,t_2}}{F_{t_2}}.$$

Since, by assumption, $B_0 > 0$ this implies that, for any $t \leq \bar{t}$

$$\xi_{B,0} = \beta^t \xi_{B,t} \Rightarrow \frac{\xi_{B,0}}{\xi_{B,t}} = \beta^t.$$

The conditions for the changes in the decentralized equilibrium (2.51) and (2.52) gives, for any $t \leq \bar{t}$

$$-\xi_{B,0} B_0' = \begin{cases} -\xi_{B,t} B_t' & \text{if } t \neq T \\ -\xi_{B,T} B_T' + \xi_{X,T} & \text{if } t = T \end{cases}$$

or

$$B_t' = \begin{cases} \beta^t B_0' & \text{if } t \neq T \\ \beta^T B_0' + \frac{\xi_{X,T}}{\xi_{B,T}} & \text{if } t = T \end{cases}.$$

Since the constraint on the total amount of available fossil fuel always binds in the decentralized equilibrium, it must be that

$$0 = \sum_{t=0}^{\infty} B'_t = \sum_{t=0}^{\bar{t}} \beta^t B'_0 + \frac{\xi_{X,T}}{\xi_{B,T}} = B'_0 \sum_{t=0}^{\bar{t}} \beta^t + \frac{\xi_{X,T}}{\xi_{B,T}} = B'_0 \frac{1-\beta^{\bar{t}+1}}{1-\beta} + \frac{\xi_{X,T}}{\xi_{B,T}}.$$

This delivers

$$B'_0 = -\frac{1-\beta}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}}$$

and

$$B'_t = \begin{cases} -\beta^t \frac{1-\beta}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} & \text{if } t \neq T \text{ and } t \leq \bar{t} \\ -\beta^t \frac{1-\beta}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} + \frac{\xi_{X,T}}{\xi_{B,T}} & \text{if } t = T \\ 0 & \text{if } t > \bar{t} \end{cases}. \tag{2.77}$$

From (2.31), the change in the climate state is

$$M'_t = \sum_{t'=0}^{t-1} \frac{dM_t}{dB_{t'}} B'_{t'} = \sum_{t'=0}^{t-1} \left(m + (1-m)(1-\delta)^{t-1-t'}\right) B'_{t'}.$$

With either of the assumptions $m = 1$ or $\delta = 0$, $m + (1-m)(1-\delta)^{t-1-t'} = 1$ and the change in the climate state becomes

$$M'_t = \sum_{t'=0}^{t-1} B'_{t'}.$$

Substituting in the expressions for $B'_t$ from (2.77) gives

$$M'_t = \begin{cases} -\frac{1-\beta^t}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} & \text{if } t \leq T \\ -\frac{1-\beta^t}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} + \frac{\xi_{X,T}}{\xi_{B,T}} & \text{if } T < t \leq \bar{t} \\ 0 & \text{if } t > \bar{t} \end{cases}.$$

The sum that is relevant for the indirect effect on welfare in the decentralized equilibrium is then

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t M'_t = \sum_{t=0}^{\bar{t}} \beta^t M'_t &= -\sum_{t=0}^{\bar{t}} \frac{\beta^t - \beta^{2t}}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} + \sum_{t=T+1}^{\bar{t}} \beta^t \frac{\xi_{X,T}}{\xi_{B,T}} \\ &= -\frac{\frac{1-\beta^{\bar{t}+1}}{1-\beta} - \frac{1-\beta^{2(\bar{t}+1)}}{1-\beta^2}}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} + \frac{1-\beta^{\bar{t}+1} - (1-\beta^{T+1})}{1-\beta} \frac{\xi_{X,T}}{\xi_{B,T}} \\ &= -\frac{\frac{1-\beta^{\bar{t}+1}}{1-\beta} - \frac{(1+\beta^{\bar{t}+1})(1-\beta^{\bar{t}+1})}{(1-\beta)(1+\beta)}}{1-\beta^{\bar{t}+1}} \frac{\xi_{X,T}}{\xi_{B,T}} + \frac{\beta^{T+1} - \beta^{\bar{t}+1}}{1-\beta} \frac{\xi_{X,T}}{\xi_{B,T}} \\ &= \frac{\beta}{1-\beta} \frac{\beta^T(1+\beta) - (1+\beta^{\bar{t}+1})}{1+\beta} \frac{\xi_{X,T}}{\xi_{B,T}}. \end{aligned}$$

The first ratio in the last expression is positive. Assuming that $\beta^T(1+\beta) > 1$, the second ratio can be either positive or negative depending on $\bar{t}$. It is negative if $\bar{t} = T+1$ and positive if $\bar{t}$ is large enough.

The conclusion from this analysis is that even in this much simplified functional-form example there is no simple relationship between the change in $X_T$ and the welfare gains from taxation.

## 2.5 Model with capital and $\sigma_Y = \theta = 1$

Above, in propositions 2.4 and 2.6, it was shown that assuming logarithmic utility simplifies the analysis. In this section I will show that if $\sigma_Y = 1$ and capital depreciates fully between periods, then most results that hold for logarithmic utility also hold in a model with capital.[5]

To begin with, when $\sigma_Y = 1$ energy and other inputs are combined into final goods production according to a Cobb-Douglas production function. It does then not make sense to distinguish between TFP, $A_Y$, the productivity of the non-energy inputs ($A_L$ or $A_{KL}$) and the productivity of the composite energy good, $A_E$: all these terms can be multiplied together to give a new, all-inclusive productivity factor. From tables 2.1 and 2.2 with $\sigma_Y = 1$, the sign of the effects of changes in these factors depend only on the sign of $\theta - 1$. If $\theta = 1$ the effects of all these technology factors on fossil-fuel use are zero. This is as before, because if $\theta = 1$, then the effects on marginal utility of consumption and marginal productivity of fossil fuel exactly balance each other.

The signs of the effects of $A_S$, $S$ and $B$ depend on the sign of $\sigma_E - 1$. If $\sigma_E > 1$, the effects, without externalities, are such that an increase in $A_S S$ or a decrease in $A_B$ decreases fossil-fuel use in the period where the change occurs and increases fossil-fuel use in all other periods (and the other way around).

So assume now that $\theta = \sigma_Y = 1$ and that capital depreciates fully between periods. Production can then be written

$$Y = D(M) A_Y \left(L^{1-\alpha} K^{\alpha}\right)^{\gamma_{KL}} Y_E^{\gamma_E},$$

where

$$Y_E = \left[\gamma_B (A_B B)^{\frac{\sigma-1}{\sigma}} + \gamma_S (A_S S)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

Define

$$\tilde{\alpha} = \alpha \gamma_{KL}$$

and let

$$F(K, B; L, S, A_Y, A_B, A_S) = A_Y L^{(1-\alpha)\gamma_{KL}} K^{\tilde{\alpha}} Y_E^{\gamma_E},$$

[5] The results that may not apply are those related to the welfare effects in section 2.4.5 and those with endogenous supply of the alternative-energy input in section 2.6.

I will now show that the equilibrium condition for fossil-fuel use in the decentralized equilibrium and the optimality condition for fossil-fuel use in the planner solution will be the same under these assumptions as in the log utility case of the model without capital (in sections 2.4.2 and 2.4.3, respectively).

### 2.5.1 Decentralized equilibrium with a fossil-fuel tax

The main difference here compared to section 2.4.2 is that the households now hold capital. This means that the households receive capital income. In addition, the rental rate of capital must also be specified; it will be denoted $r_t$. Since the model is deterministic and there is full depreciation, the rental rate of capital will also be the interest rate used for the discounting of future profits received by fossil-fuel resource owners.

This gives the households' budget constraint

$$C_t + K_{t+1} = w_t L + r_t K_t + \pi_{B,t} + \pi_{S,t} + g_t. \tag{2.78}$$

Apart from these changes, the definition of the decentralized equilibrium is the same as without capital.

A decentralized equilibrium with taxation consists of sequences of quantities $\{B_t, C_t, K_{t+1}\}_{t=0}^{\infty}$, fossil-fuel taxes $\{\tau_t\}_{t=0}^{\infty}$, lump-sum tax rebates $\{g_t\}_{t=0}^{\infty}$ and prices $\{p_{B,t}, p_{S,t}, r_{t+1}, w_t\}_{t=0}^{\infty}$ such that

- Households choose their consumption and investments to maximize their discounted utility $\sum_{t=0}^{\infty} \beta^t u(C_t)$ subject to the budget constraint (2.78) for all $t$.
- Prices are competitively determined.
- Fossil-fuel extracting firms maximize the discounted profit from extraction.
- The government budget is balanced in each period.

As before, I assume that the alternative-energy input and labor are both inelastically supplied so that their quantities are not endogenously determined. Their prices will, however, be endogenously determined and affect the profits of the suppliers of the alternative-energy input (which in the end goes to the households) and the labor income of the households.

I will now derive the implications of each of these conditions in turn and then characterize the resulting equilibrium allocation.

**Households** The utility maximization problem of the representative household is[6]

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t) \text{ s.t. } (2.78).$$

By substituting for consumption from the budget constraint, the decision problem is reduced to choosing next period's capital at all times. Taking the first-order condition with respect to $K_{t+1}$, one obtains

$$\beta^t U'(C_t) = \beta^{t+1} U'(C_{t+1}) r_{t+1} \Rightarrow \frac{1}{\beta}\frac{C_{t+1}}{C_t} = r_{t+1}. \tag{2.79}$$

**Competitive prices** When final goods producers act as price takers, the equilibrium prices are

$$\begin{aligned} p_{B,t} &= D(M_t)F_{B,t},\ p_{S,t} = D(M_t)F_{S,t} \\ w_t &= D(M_t)F_{L,t},\ r_t = D(M_t)F_{K,t} = \tilde{\alpha}\tfrac{Y_t}{K_t} \end{aligned} \tag{2.80}$$

**Firms supplying fossil fuel** As in the case without capital, maximization of discounted profits from fossil-fuel extraction requires that the discounted after-tax price of fossil fuel is the same in all periods with positive fossil-fuel use. The profits are discounted using the rental rate of capital. Since there are no extraction costs for fossil fuel, the profits from fossil-fuel extraction is the after tax price of fossil fuel times the extracted quantity. So the situation is exactly the same as in the case without capital and, consequently, the profit maximization condition is the same as without capital (2.46). That is,

$$\left[\prod_{t'=0}^{t-1} \frac{1}{r_{t'+1}}\right](1-\tau_t)p_{B,t}$$

is the same in all periods with positive aggregate fossil-fuel use, $B_t > 0$.

**Balanced government budget** The government balances its budget in each time period implying that, for each $t$,

$$g_t = \tau_t p_{B,t} = \{(2.80)\} = \tau_t D(M_t) F_{B,t}.$$

[6]Here only the capital investment decision is included. In principle trade in the shares in energy companies could be included but in equilibrium there will be no trade in these shares. Including these shares would simply allow us to compute their equilibrium prices.

**Equilibrium allocation** Since there are no extraction costs for the energy inputs, the prices $p_{B,t}$ and $p_{S,t}$ directly gives the profits of the energy supplying firms

$$\begin{aligned}\pi_{B,t} &= (1-\tau_t)p_{B,t}B_t = (1-\tau_t)D(M_t)F_{B,t}B_t \\ \pi_{S,t} &= p_{S,t}S_t = D(M_t)F_{S,t}S_t.\end{aligned}$$

Summing up the income sources for the households

$$\begin{aligned}w_t l_t + r_t K_t + \pi_{B,t} + \pi_{S,t} + T_t &= D(M_t)F_{L,t}L_t + D(M_t)F_{K,t}K_t \\ &\quad + D(M_t)F_{B,t}B_t + D(M_t)F_{S,t}S_t \\ &= D(M_t)F_t = Y_t,\end{aligned}$$

where the last step follows since $F$ has constant returns to scale in $L$, $K$, $B$ and $S$.

Substituting the capital rental rate from (2.80) in the households' first-order condition (2.79) delivers

$$\frac{C_{t+1}}{C_t} = \beta\tilde{\alpha}\frac{Y_{t+1}}{K_{t+1}}.$$

This condition is fulfilled by the consumption/investment rule

$$C_t = (1-\tilde{\alpha}\beta)Y_t \text{ and } K_{t+1} = \tilde{\alpha}\beta Y_t.$$

The discount factor for profits in period $t \geq 1$ is

$$\prod_{t'=0}^{t-1}\frac{1}{r_{t'+1}} = \prod_{t'=0}^{t-1}\frac{1}{\tilde{\alpha}}\frac{K_{t'+1}}{Y_{t'+1}} = \prod_{t'=0}^{t-1}\frac{1}{\tilde{\alpha}}\frac{\tilde{\alpha}\beta Y_{t'}}{Y_{t'+1}} = \prod_{t'=0}^{t-1}\frac{\beta Y_{t'}}{Y_{t'+1}} = \beta^t\frac{Y_0}{Y_t}.$$

Substituting this into the equilibrium condition gives that

$$\beta^t\frac{Y_0}{Y_t}(1-\tau_t)p_{B,t} = \beta^t\frac{Y_0}{Y_t}(1-\tau_t)D(M_t)F_{B,t}$$

should be the same for all periods with positive fossil-fuel use.

Comparing two time periods $t_1$ and $t_2$ such that $B_{t_1} > 0$ and $B_{t_2} > 0$, we deduce that

$$\beta^{t_1}(1-\tau_{t_1})\frac{F_{B,t_1}}{F_{t_1}} = \beta^{t_2}(1-\tau_{t_2})\frac{F_{B,t_2}}{F_{t_2}}. \tag{2.81}$$

The ratio $\frac{F_{B,t}}{F_t}$ is independent of the capital stock. This completes the characterization of the competitive equilibrium.

Comparing the condition for fossil-fuel use here to the corresponding expressions in the model without capital, given by equations (2.5) and (2.48), it can be seen that the expressions are the same. With Cobb-Douglas production, capital cancels from expression (2.81).

This implies that fossil-fuel use will be the same here as in the model in section 2.4.2 without capital and with logarithmic utility.

### 2.5.2 Planner solution

The planner solves the problem

$$\max_{\{B_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(D(M_t)F_t - K_{t+1}) \text{ s.t.} \sum_{t=0}^{\infty} B_t \leq Q.$$

The first-order condition with respect to $K_{t+1}$ is

$$\beta^t U'(C_t) = \beta^{t+1} U'(C_{t+1}) F_{K,t+1}.$$

With the functions used here this equation delivers

$$\frac{1}{C_t} = \beta \frac{1}{C_{t+1}} \tilde{\alpha} \frac{Y_{t+1}}{K_{t+1}} \Rightarrow \frac{K_{t+1}}{C_t} = \tilde{\alpha}\beta \frac{Y_{t+1}}{C_{t+1}}.$$

This is fulfilled by having

$$C_t = (1 - \tilde{\alpha}\beta)F_t \text{ and } K_{t+1} = \tilde{\alpha}\beta F_t \ \forall t,$$

which is the same as in the decentralized equilibrium.

The first-order condition with respect to $B_t$, for any $t$ such that $B_t > 0$, gives

$$\lambda = \beta^t U'(C_t) D(M_t) F_{B,t} + \sum_{t'=t+1}^{\infty} \beta^{t'} U'(C_{t'}) F_{t'} D'(M_{t'}) \frac{dM_{t'}}{dB_t},$$

where $\lambda$ is the multiplier on the constraint on the total supply of fossil fuel. With the functional forms assumed here, we obtain that for any $t$ such that $B_t > 0$

$$\beta^t \frac{F_{B,t}}{F_t} + \sum_{t'=t+1}^{\infty} \beta^{t'} \frac{D'_{t'}}{D_{t'}} \frac{dM_{t'}}{dB_t} = \lambda. \tag{2.82}$$

Comparing conditions (2.82) and (2.56) with $\theta = 1$, we note the same relative comparison between time periods. As in the decentralized equilibrium, capital cancels out from this expression.

In conclusion, fossil-fuel use will be the same here as in the model in section 2.4.3 without capital and with logarithmic utility.

### 2.5.3 Optimal taxation

Comparing the competitive equilibrium with taxation and the planner solution, the investment choices are the same in both cases in the sense that the same share of final good production is invested in both cases.

This means that a tax system that induces the optimal path of fossil-fuel use in the decentralized equilibrium will induce the social optimum.

Comparing the equilibrium condition in the decentralized equilibrium with taxation (2.81) to the optimality condition from the planner solution (2.82), the optimum can be implemented by a tax system that fulfills

$$\tau_t \frac{F_{B,t}}{F_t} = - \sum_{t'=t+1}^{\infty} \beta^{t'-t} \frac{D'_{t'}}{D_{t'}} \frac{dM_{t'}}{dB_t}$$

Comparing this to (2.73) with $\theta = 1$, it can be seen that they are equivalent. Therefore, the optimal taxes will be the same here as in section 2.4.4 without capital and with logarithmic utility.

## 2.6 Elastic supply of the alternative-energy input

So far, the alternative-energy input, $S$, has been supplied inelastically and using alternative energy has not been associated with any costs. In this section I will consider the case where the provision of $S$ uses resources that could be used elsewhere. To study this I will use a model without capital and without externalities. In my setting, two different types of costs are conceivable. One is that extraction of the alternative-energy input requires labor. Another one would be that the use of the alternative-energy input requires some consumption of final goods. I will here focus on the case where the alternative-energy input uses labor.

I assume that one unit of the alternative-energy input uses $a_S$ units of labor. Let the total amount of available labor be exogenously given and denoted by $\bar{L}$. Labor can then either be used to produce the alternative-energy input or to produce final goods. Let the amount of labor used in alternative-energy production and final-good production be $L^S$ and $L^Y$, respectively. These must fulfill $L^S + L^Y = \bar{L}$. The alternative-energy input is now given by the linear production schedule $S = a_S L^S$. I will assume that labor can move freely between the sectors so that the labor allocation decision can be treated as a static decision made within each period.

Production in period $t$ depends on the amount of fossil fuel used, $B_t$, the amount of labor used in final good production, $L_t^Y$, the amount of the alternative-energy input, $S_t$, and on the set of exogenously given variables $\Gamma_t$ which now consists of the productivities $(A_{Y,t}, A_{L,t}, A_{E,t}, A_{B,t}, A_{S,t})$, the productivity of labor in producing the alternative-energy input, $a_{S,t}$, and the total amount of labor $\bar{L}$. So, production, which is equal to consumption, can be written as

$$C_t = Y_t = F(B_t, L_t^Y, S_t; \Gamma_t). \tag{2.83}$$

The production function is the same as in section 2.2 and defined in equations (2.11) and (2.12).

Since I am ruling out any externalities, the planner solution and the decentralized equilibrium will coincide and I will only solve for the planner solution.

### 2.6.1 Planner solution

In each period, the planner chooses how much fossil fuel to use, how much of the alternative-energy input to use and how to allocate labor between alternative-energy provision and final-goods production. The constraints are the total supply of fossil fuel, the constraint on total available labor in each period, the production function for the alternative-energy input and non-negativity constraints on all the chosen variables. This gives the following planner problem:

$$
\begin{aligned}
\max_{\{B_t, S_t, L_t^Y, L_t^S\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} U\left(F(B_t, L_t^Y, S_t; \Gamma_t)\right) \\
\text{s.t.} & \sum_{t=0}^{\infty} B_t \leq Q_0 \\
& \forall t\text{: } L_t^Y + L_t^S \leq \bar{L}_t,\ S_t = a_{S,t} L_t^S \\
& \forall t\text{: } ,\ L_t^Y \geq 0,\ L_t^S \geq 0,\ B_t \geq 0.
\end{aligned}
$$

The Lagrangian of this problem is

$$
\begin{aligned}
\mathcal{L} = & \sum_{t=0}^{\infty} \beta^t U\left(F(B_t, L_t^Y, S_t; \Gamma_t)\right) + \lambda \left[Q_0 - \sum_{t=0}^{\infty} B_t\right] \\
& + \sum_{t=0}^{\infty} \mu_{L,t} \left[\bar{L}_t - L_t^Y - L_t^S\right] + \sum_{t=0}^{\infty} \mu_{S,t} \left[a_{S,t} L_t^S - S_t\right] \\
& + \sum_{t=0}^{\infty} \left[\eta_{Y,t} L_t^Y + \eta_{S,t} L_t^S + \eta_{B,t} B_t\right]
\end{aligned}
$$

Taking first-order conditions, we obtain

$$
\begin{aligned}
B_t &: \beta^t U'(C_t) F_{B,t} = \lambda - \eta_{B,t} \\
S_t &: \beta^t U'(C_t) F_{S,t} = \mu_{S,t} \\
L_t^Y &: \beta^t U'(C_t) F_{L,t} = \mu_{L,t} - \eta_{L,t} \\
L_t^S &: \mu_{L,t} = \mu_{S,t} a_{S,t} + \eta_{S,t}.
\end{aligned}
$$

I will now assume that $\sigma_E < \infty$, that is, that the energy sources are not perfect substitutes. This will imply that the production function fulfills Inada conditions for $B$, $S$ and $L^Y$ so that non-negativity constraints will never bind. That is $\eta_{Y,t} = \eta_{S,t} = \eta_{B,t} = 0$. Furthermore, the marginal product of all inputs will always be strictly positive implying that the inequality constraints will hold with equality. The first-order conditions with respect to $S_t$, $L_t^Y$ and $L_t^S$ can now be combined to give

$$F_{L,t} = a_{S,t} F_{S,t}, \tag{2.84}$$

which means that the marginal product of labor must be the same in both possible uses in all periods.

Since $B_t > 0$ for all $t$, the first order condition with respect to $B_t$ implies that, for any $t_1$ and $t_2$,

$$\beta^{t_1} U'(F_{t_1}) F_{B,t_1} = \beta^{t_2} U'(F_{t_2}) F_{B,t_2},$$

which, as before, says that the marginal value of fossil-fuel use should be the same in all periods. For a given $t$, $U'(C_t)F_{B,t}$ will depend on the same variables as appear in the production function, that is, $B_t$, $L_t^Y$, $S_t = a_{S,t} L_t^S$ and $\Gamma_t$. Along the same lines as before it is now possible to define

$$H(B_t, L_t^Y, a_{S,t} L_t^S; \Gamma_t) = U'(F_t) F_{B,t}.$$

However, since (2.84) determines the intratemporal allocation of labor between the sectors, and implicitly defines $L_t^Y$ and $L_t^S$ as functions of $B_t$ and $\Gamma_t$, $H$ can be written as a function of only $B_t$ and $\Gamma_t$. So I will define

$$\begin{aligned} \tilde{H}(B_t; \Gamma_t) &= H(B_t, L_t^Y, a_{S,t} L_t^S; \Gamma_t) \\ &\text{where} \quad L_t^Y + L_t^S = \bar{L}_t \text{ and (2.84) is fulfilled.} \end{aligned} \tag{2.85}$$

Using this definition, the equilibrium condition, for any $t_1$ and $t_2$ is

$$\beta^{t_1} \tilde{H}_{t_1} = \beta^{t_2} \tilde{H}_{t_2} \tag{2.86}$$

## 2.6.2 Changes in $\{\Gamma_t\}_{t=0}^{\infty}$

The equilibrium condition is now very similar to (2.6) and the effects of varying the exogenous variables in $\Gamma_T$ can be analyzed in much the same way. The main difference is that when differentiating $\tilde{H}$ with respect to either of its arguments, the effect on the labor allocation must also be taken into account. I will now calculate the effect on $\tilde{H}_t$ of changing $B_t$.

Firstly, to find the effect on the labor allocation decision, I will differentiate condition (2.84) with respect to $B_t$ and treat $L_t^S$ and $L_t^Y$ as functions of $B_t$ (and $\Gamma_t$). Since all variables are in the same time period, I will suppress the time indices. Differentiating both sides of (2.84) with respect to $B$ gives

$$a_S\left[F_{SB}+F_{SS}a_SL_B^S-F_{SL}L_B^S\right]=F_{LB}+F_{LS}a_SL_B^S-F_{LL}L_B^S,$$

where I have used that $L_B^Y=-L_B^S$. Solving for $L_B^S$ gives

$$L_B^S=\frac{a_SF_{SB}-F_{LB}}{2a_SF_{LS}-F_{LL}-a_S^2F_{SS}}. \tag{2.87}$$

When differentiating $\tilde{H}$ with respect to $B_t$, all time indices are the same and I will suppress them. The derivative with respect to $B$ is then

$$\begin{aligned}\frac{\partial\tilde{H}}{\partial B}=&U''(F)F_B\left[F_B+F_Sa_SL_B^S-F_LL_B^S\right]\\&+U'(F)\left[F_{BB}+F_{BS}a_SL_B^S-F_{BL}L_B^S\right]\\=&U'(C)\left[F_{BB}-\theta\frac{F_B^2}{F}+(F_{BS}a_S-F_{BL})L_B^S\right].\end{aligned}$$

The optimality condition (2.84) was used to cancel terms from the first parenthesis and the form of the utility function (2.10) was used to substitute for $U''(C)$.

Substituting $L_B^S$ from (2.87) gives that

$$\frac{\partial\tilde{H}}{\partial B}=U'(C)\left[F_{BB}-\theta\frac{F_B^2}{F}+\frac{(F_{BS}a_S-F_{BL})^2}{2a_SF_{LS}-F_{LL}-a_S^2F_{SS}}\right]. \tag{2.88}$$

It can be shown that this derivative is negative (see equation (2.102) in appendix 2.A.4). This leads to the following proposition.

**Proposition 2.9.** *If $\sigma_Y>0$ and $\sigma_E<\infty$, then $B_t>0$ for all $t$, the labor allocation is interior for all $t$ and the effects on fossil-fuel use of a change in $X_T$ is*

$$\begin{aligned}\operatorname{Sgn}\left(\frac{dB_T}{dX_T}\right)=&\operatorname{Sgn}\left(\frac{\partial\tilde{H}_T}{\partial X_T}\right)\quad \textit{and}\\\operatorname{Sgn}\left(\frac{dB_t}{dX_T}\right)=&\operatorname{Sgn}\left(-\frac{\partial\tilde{H}_T}{\partial X_T}\right)\quad \textit{for all } t\neq T\end{aligned}$$

*Proof.* When $\sigma_Y>0$ and $\sigma_E<\infty$, the production function fulfills Inada conditions for all its inputs and there will be an interior solution in all periods. Starting from the equilibrium condition (2.86) and the observation that $\frac{\partial\tilde{H}_t}{\partial B_t}<0$ for all $t$, the rest of the proposition follows from the same logic as the proof of proposition 2.1. □

In conclusion, the effect of a change in the technology factor $X_T$ depends on the partial effect of the change on $\tilde{H}_T$. Since all variables considered in what follows, will concern period $T$, I will suppress the time indices from the notation.

To begin with, the effect of the change in $X$ on the labor allocation must be found. Since $a_S$ appears explicitly in expression (2.84), differentiating with respect to $a_s$ is qualitatively different than differentiating with respect to any of the other technology factors. I will start by looking at changes in $X \in \{A_Y, A_L, A_E, A_B, A_S\}$. Taking the dependency of $L^S$ and $L^Y$ on $X$ into account and differentiating both sides of the labor allocation condition (2.84) with respect to $X$ gives

$$a_S \left[F_{SX} + F_{SS} a_S L_X^S - F_{SL} L_X^S\right] = F_{LX} + F_{LS} a_S L_X^S - F_{LL} L_X^S,$$

where I have used that $L_X^Y = -L_X^S$. Solving for $L_X^S$ gives

$$L_X^S = \frac{a_S F_{SX} - F_{LX}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}. \tag{2.89}$$

Differentiating $\tilde{H}$ with respect to $X$ then yields

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial X} =& U''(F) F_B \left[F_X + F_S a_S L_X^S - F_L L_X^S\right] \\ & + U'(F) \left[F_{BX} + F_{BS} a_S L_X^S - F_{BL} L_X^S\right] \\ =& U'(C) \left[F_{BX} - \theta \frac{F_B F_X}{F} + \left(F_{BS} a_S - F_{BL}\right) L_X^S\right].\end{aligned} \tag{2.90}$$

Here, the optimality condition (2.84) was used to cancel terms from the first parenthesis and the form of the utility function (2.10) was used to substitute for $U''(C)$.

Looking at the expression within the parenthesis, it consists of two parts. The first part, consisting of two terms, is the same as the expression determining the sign in the case with inelastic supply of the clean energy input (2.17). The second part relates to the effect of the change in the allocation of labor. An increase in $L^S$ increases the amount of the alternative-energy input and decreases the amount of labor used in final goods production. The derivatives are (see appendix 2.A.4)[7]

$$\begin{aligned}F_{BS} =& F_B \left[\frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L}\right] \frac{G_S}{G_B + G_S} \frac{1}{S} \\ F_{BL} =& F_B \frac{1}{\sigma_Y} \frac{G_L}{G_L + G_E} \frac{1}{L^Y}.\end{aligned}$$

[7] In all the following calculations, the $G$s are defined as in (2.16).

The first derivative is ambiguous since it depends on the size of $\frac{G_L}{G_E+G_L}$ The positive part of this comes from the fact that an increase in the amount of alternative energy increases the productivity of fossil fuel in producing the composite energy good, while the negative part comes from the fact that there is more energy in relation to the complementary input labor, which decreases the marginal product of energy. The second derivative is positive indicating that an increase in the amount of labor used in final goods production increases the marginal product of fossil fuel. The combination of the derivatives is (see appendix 2.A.4)

$$a_S F_{BS} - F_{BL} = F_B \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{G_S}{G_B + G_S} \frac{a_S}{S}.$$

This expression is negative under assumption (2.13). That is, the total effect of moving labor from final good production to production of the alternative-energy input is to decrease the marginal product of fossil fuel.

From the calculations in the appendix (see 2.A.4) it follows that $L^S_{A_Y} = 0$, $L^S_{A_L} \geq 0$, $L^S_{A_E} \leq 0$ and $L^S_{A_B} < 0$ while the sign of $L^S_{A_S}$ is ambiguous. As long as $F_{BX} - \theta \frac{F_B F_X}{F}$ and $L^S_X$ has opposite signs, the sign of $\frac{\partial \tilde{H}}{\partial X}$ is unambiguously determined. Combining table 2.1 with the signs of $L^S_X$, it follows that, under assumptions (2.13)-(2.15)

$$\frac{\partial \tilde{H}}{\partial A_Y} \leq 0.$$

For $A_B$, the sign of $F_{BA_B} - \theta \frac{F_B F_{A_B}}{F}$ is ambiguous and for $A_S$, the sign of $L^S_{A_S}$ is ambiguous. So for all these factors the partial derivative of $\tilde{H}$ must be calculated.

Substituting $L^S_X$ from (2.89) into (2.90) gives

$$\frac{\partial \tilde{H}}{\partial X} = U'(C) \left[ F_{BX} - \theta \frac{F_B F_X}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SX} - F_{LX})}{2 a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} \right]. \tag{2.91}$$

The calculation of these derivatives can be found in the appendix (see 2.A.4). The derivatives that are unambiguously determined by assumptions (2.13)-(2.15) are

$$\frac{\partial \tilde{H}}{\partial A_Y} \leq 0, \ \frac{\partial \tilde{H}}{\partial A_E} < 0 \text{ and } \frac{\partial \tilde{H}}{\partial A_S} < 0 \tag{2.92}$$

Considering a change in $A_L$, the derivative of $\tilde{H}$ with respect to $A_L$ is

(see appendix 2.A.4 for more detail)

$$\frac{\partial \tilde{H}}{\partial A_L} = u'(C)\frac{F_B}{A_L}\frac{\frac{1}{\sigma_Y}\left(1+\frac{1-\sigma_Y}{\sigma_E}-\theta\right)\frac{G_S}{G_B+G_S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S}+\frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S}+\frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{a_S}{S}$$
$$+u'(C)\frac{F_B}{A_L}\frac{\left(\frac{1}{\sigma_Y}-\theta\right)\frac{1}{\sigma_E}\frac{G_L}{G_E+G_L}\frac{G_B}{G_B+G_S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S}+\frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S}+\frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{a_S}{S}$$

The sign of this expression is, in general, ambiguous, reflecting the two opposing effects. If $\theta \leq 1+\frac{1-\sigma_Y}{\sigma_E} > 1$ it is unambiguously positive. If $\theta > 1+\frac{1-\sigma_Y}{\sigma_E}$, the first term is negative while the second term is positive.

Considering a change in $A_B$, the derivative of $\tilde{H}$ with respect to $A_S$ becomes (see appendix 2.A.4 for more detail)

$$\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_B} = \frac{F_B}{A_B}\frac{\sigma_E-1}{\sigma_E}\frac{G_S}{G_B+G_S}$$
$$-\frac{F_B}{A_B}\left(\frac{1-\sigma_Y}{\sigma_Y}\frac{G_L}{G_E+G_L}+(\theta-1)\frac{G_E}{G_E+G_L}\right)\frac{G_B}{G_B+G_S}$$
$$+\frac{F_B}{A_B}\frac{\left(\frac{1}{\sigma_E}-\frac{1}{\sigma_Y}\right)^2\frac{G_B}{G_B+G_S}\frac{G_S}{G_B+G_S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S}+\frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S}+\frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{a_S}{S}.$$

The first two terms are the same as in the case with inelastic supply of $S$. The third term is positive. If $G_B << G_S$ or $G_B >> G_S$ the third term is small and the results that the derivative is positive if $G_S >> G_B$ and negative if $G_B << G_S$ holds also with elastic supply of $S$.

I will now turn to the effects of a change in $a_S$. Differentiating condition (2.84) with respect to $a_S$ gives

$$F_{LS}\left(a_S L^S_{a_S}+L^S\right)-F_{LL}L^S_{a_S} = F_S + a_S\left(a_S F_{SS}L^S_{a_S}-F_{SL}L^S_{a_S}+L^S F_{SS}\right),$$

where I have used that $L^Y_{a_S} = -L^S_{a_S}$. Solving for $L^S_{a_S}$ gives

$$L^S_{a_S} = \frac{F_S + a_S L^S F_{SS} - L^S F_{LS}}{2a_S F_{LS} - a_S^2 F_{SS} - F_{LL}}. \tag{2.93}$$

A change in $a_S$ affects the marginal value of fossil-fuel use both by affecting the amount of alternative energy available, for a given amount of labor used in the production of the alternative-energy input, and by affecting the allocation of labor. Differentiating $\tilde{H}$ with respect to $a_S$

gives

$$\frac{\partial \tilde{H}}{\partial a_S} = U''(F)\left[L^S F_S - L^S_{a_S} F_L + a_S F_S L^S_{a_S}\right] F_B$$
$$+U'(F)\left[L^S F_{BS} - L^S_{a_S} F_{BL} + a_S F_{BS} L^S_{a_S}\right]$$
$$= U'(F)\left[\left(F_{BS} - \theta \frac{F_B F_S}{F}\right) L^S + \left(a_S F_{BS} - F_{BL}\right) L^S_{a_S}\right],$$

where the labor allocation condition (2.84) and the shape of the utility function (2.10) were used. Substituting for $L^S_{a_S}$ gives

$$\frac{\partial \tilde{H}}{\partial a_S} = U'(F)\left(F_{BS} - \theta \frac{F_B F_S}{F}\right) L^S$$
$$+U'(F)\frac{\left(F_S + a_S L^S F_{SS} - L^S F_{LS}\right)\left(a_S F_{BS} - F_{BL}\right)}{2a_S F_{LS} - a_S^2 F_{SS} - F_{LL}}. \quad (2.94)$$

As shown in appendix 2.A.4, this expression is negative under assumptions (2.13)-(2.15).

This can all be summarized in the following proposition:

**Proposition 2.10.** *Under the assumptions of proposition 2.9 and assumptions* (2.13)-(2.15) *and for* $t \neq T$*,*

$$\frac{dB_T}{dX_T} \leq 0 \text{ and } \frac{dB_t}{dX_T} \geq 0$$

*for* $X \in \{A_Y, A_E, A_S, a_S\}$ *while the effect of varying* $A_L$ *or* $A_B$ *is ambiguous.*

Thus, compared to the case with inelastic supply of the alternative-energy input, the effects are the same except for the fact that the effect of the labor augmenting technology $A_L$ is now ambiguous. Furthermore, the effects of varying $a_s$ on fossil-fuel use have the same sign as the effects of varying $A_S$.

## 2.7 Discussion

In this chapter I have investigated how the trends of technological development affects the intertemporal pattern of fossil-fuel use. I have also considered the robustness of the results to a number of variations of the assumptions. Apart from the results regarding the technology that is complementary to energy, in the case where the complementary input could also be used to produce alternative energy, the qualitative results have remained the same throughout the analysis. In this section, I will discuss four different aspects of the assumptions I have made. I will first briefly discuss three assumptions that I have made and that I do not think affect the results much. I will then discuss the assumption about zero extraction costs of fossil fuel in somewhat more detail. In particular, I will show, using a simple example, that this assumption does seem to be important.

Throughout, I have treated technological developments as exogenous. In reality, they are driven by forward-looking decisions. Endogenizing technology could potentially provide interesting insights regarding the interaction between the fossil-fuel supply and forward-looking research activity. In such a model, subsidies to research on the different types of technology could be studied instead of exogenous changes in the technology factors. However, while the interaction between the different decisions made could amplify or damped the mechanisms studied here, it does not seem likely that the signs of the effects should change.

Except for the anlyses in sections 2.3 and 2.5, the treatment here abstracts from capital accumulation. While the omission simplifies the analysis, I do not think that it affects the qualitative results much. The two sections where I do include capital (but restrict the models in other ways) confirm the results from the other sections. In section 2.5 I also provide results for an infinite time horizon model with $\theta = \sigma_Y = 1$. More realistic assumptions would probably involve a $\theta$ that is a little bit larger than one, while $\sigma_Y$ would be lot smaller. From both tables 2.1 and 2.2 it can be seen that changes away from the assumptions of section 2.5 toward more realistic values should strengthen the effects I derive. Van der Ploeg and Withagen (2011) also find a Weak Green Paradox in a growth model with capital.

In section 2.4 I consider different specifications of the description of the carbon cycle. There I derive results under somewhat restrictive assumptions. While it may be difficult to generalize the analysis analytically, assumptions, the proofs rely on a sequence of sufficient conditions. This suggests that the results may well apply under much more general assumptions. In particular, for most of the results, I assumed that either $\delta = 0$ or that $m = 1$. The implications of these assumptions that are most important for the results are that they imply that the climate state is increasing over time and that changes in emissions accumulate over time. In reality, the climate state will keep increasing for a long time. So that part seems unproblematic. It also seems reasonable that, if comparing two different emission paths where one has higher emissions for a couple of decades, the difference during this time between the induced climate paths should also increase over time.

A potentially more critical assumption throughout my analysis is that fossil fuel is costlessly extracted from a given total supply. Extraction decisions are then based completely on the relative profitability (or marginal value in planner solutions) between extraction at different points in time. This means that the scarcity value of the resource is very important. This may not be too bad an assumption for oil, but the supply decisions for coal are more likely to be determined by extraction costs than by resource scarcity. My assumption was made to emphasize the mechanisms studied here. That is, the way I model fossil-fuel supply highlights the way that scarcity rents influence the results. It is, however, not an unproblematic assumption. See Hart and Spiro (2011) for a critical discussion of the role of the scarcity rent in the price of oil and coal. Having extraction costs for fossil fuel that increase as the remaining resources decrease has been studied in the Green Paradox literature (see, e.g., van der Werf and Di Maria (2011)). The typical result there is that a decrease in the future value of fossil-fuel use results in more fossil-fuel use in the short run but less total fossil-fuel use. In the terminology of Gerlagh (2011), this means that there is still a Weak Green Paradox but that the existence of a Strong Green Paradox is less obvious.

Smulders et al. (2010) find that a green paradox can arise also without scarcity. They, however, consider the effects of announcing a future tax on fossil-fuel use rather than an improvement in the future state of alternative-energy technology. The results there are driven by the fact that the reduction in future production, caused by taxation, increases the value of investments, which in turn induces more fossil-fuel use in the short run to make investments. When considering an improvement in the future state of technology, this investment effect should go in the other direction. I will now demonstrate this using a simple example.

The assumption of zero extraction costs is on one of the extreme ends of the spectrum of possible assumptions in the sense that only scarcity, and not extraction costs, matters. I will now go to the other extreme and assume that scarcity plays no role at all and that extraction of fossil fuel is associated with extraction costs that are independent of the remaining stock. I will keep the model as simple as possible. In order to capture the involved dynamics, I need to have capital in the model and I need (at least) three time periods. I will not include labor. I will assume that capital can be used for three different activities: directly in final goods production, in extraction of fossil fuel and in generation of alternative energy. I will assume that both fossil-fuel extraction and alternative-energy generation are linear in the amount of capital used. The only technology factor that I will consider is the technology for alternative-energy generation. I will assume that capital can be reallocated freely within a period so that the allocation decision is static. Let the total amount of capital, in a period, be $K$ and the amount used in the different sectors be $K_Y$, $K_B$ and $K_S$, respectively.

Production in a period is given by

$$Y = F_Y(K_Y, E) = \left[K_Y^{\frac{\sigma_Y - 1}{\sigma_Y}} + E^{\frac{\sigma_Y - 1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y - 1}},$$

where

$$E = F_E(B, S) = \left[B^{\frac{\sigma_E - 1}{\sigma_E}} + S^{\frac{\sigma_E - 1}{\sigma_E}}\right]^{\frac{\sigma_E}{\sigma_E - 1}}$$

and

$$B = K_B,\; S = A_S K_S.$$

The division of $K$ into $K_Y$, $K_B$ and $K_S$ can be shown to be

$$K_Y = \frac{\left[1 + A_S^{\sigma_E - 1}\right]^{\frac{\sigma_E - \sigma_Y}{\sigma_E - 1}}}{1 + A_S^{\sigma_E - 1} + \left[1 + A_S^{\sigma_E - 1}\right]^{\frac{\sigma_E - \sigma_Y}{\sigma_E - 1}}} K$$

$$K_B = \frac{1}{1 + A_S^{\sigma_E - 1} + \left[1 + A_S^{\sigma_E - 1}\right]^{\frac{\sigma_E - \sigma_Y}{\sigma_E - 1}}} K$$

$$K_S = \frac{A_S^{\sigma_E - 1}}{1 + A_S^{\sigma_E - 1} + \left[1 + A_S^{\sigma_E - 1}\right]^{\frac{\sigma_E - \sigma_Y}{\sigma_E - 1}}} K.$$

The important observation here is that

$$\frac{\partial B}{\partial K} = \frac{\partial K_B}{\partial K} > 0 \text{ and } \frac{\partial B}{\partial A_S} = \frac{\partial K_B}{\partial A_S} < 0. \tag{2.95}$$

That is, the more capital there is, the more capital is used for fossil-fuel extraction and therefore the higher is fossil-fuel use. Also, the better is the technology for alternative-energy generation, the less capital is used for fossil-fuel extraction and, therefore, the lower will fossil-fuel use be.

Final good production can then be shown to be

$$Y = \left[1 + \left(1 + A_S^{\sigma_E - 1}\right)^{\frac{\sigma_Y - 1}{\sigma_E - 1}}\right]^{\frac{1}{\sigma_Y - 1}} K \equiv \hat{F}(A_S) K,$$

where

$$\hat{F}(A_S) = \left[1 + \left(1 + A_S^{\sigma_E - 1}\right)^{\frac{\sigma_Y - 1}{\sigma_E - 1}}\right]^{\frac{1}{\sigma_Y - 1}} \text{ and } \hat{F}'(A_S) > 0.$$

I will only be interested in varying $A_{S,3}$. I will therefore set $\hat{F}(A_{S,1}) = \hat{F}(A_{S,2}) = \hat{F}$.

Assuming full depreciation of capital, the intertemporal optimization problem is now

$$\max_{K_2,K_3} U\left(\hat{F}K_1 - K_2\right) + \beta U\left(\hat{F}K_2 - K_3\right) + \beta^2 U\left(\hat{F}(A_{S,3})K_3\right).$$

The first order conditions are

$$\begin{aligned} K_2 &: U'(C_1) = \beta \hat{F} U'(C_2) \\ K_3 &: U'(C_2) = \beta \hat{F}(A_{S,3}) U'(C_3) = \beta \hat{F}(A_{S,3})^{1-\theta} K_3^{-\theta}. \end{aligned}$$

Consider an increase in $A_{S,3}$: it will increase the marginal product of capital and decrease the marginal utility of consumption. Assuming that $\theta > 1$, the net effect is a decrease in the marginal value of capital in period 3. In order to maintain the first-order conditions at equality, consumption has to increase in periods 1 and 2. Increasing consumption in period 1 means decreasing investment, implying that $K_2$ must decrease. If second-period consumption increases, while second-period capital decreases, then second-period investment also must decrease.

Thus, the conclusion is that if $A_{S,3}$ increases, we will observe decreases in $K_2$ and $K_3$. Using the derivatives (2.95), this implies that first-period fossil-fuel use is unchanged while second and third-period fossil-fuel use decreases. That is, the results of the green paradox do not hold here. An increase in the future state of alternative-energy technology leads to a decrease in fossil-fuel use in both the short and the long run. Since the results here rely partly on the value of investments, there should be a similar effect for an improvement in any technology factor. There will also be an effect of intratemporal reallocation of capital within the period. If the technology for fossil-fuel extraction changes, there would also be a direct effect.

## 2.8 Concluding remarks

In this chapter I have analyzed how technological trends affect the intertemporal pattern of fossil-fuel use. Throughout the chapter I have assumed that fossil fuel was costlessly extracted from a given total supply. Under that assumption, the conclusions seem to be robust.

Regarding the Green Paradox, the results derived here confirms the existence of a Weak Green Paradox, at least as long as the fossil-fuel supply is driven by scarcity. A future improvement in either technology for alternative-energy generation or energy-saving technology leads to increased fossil-fuel use in the short run.

My study also emphasizes that the developments of other technology factors, as the result of increased spending on a particular type of technology, also matter for the ultimate effects on fossil-fuel use. That is, if increased research into one type of technology crowds out research of other types of technologies, this crowding out must also be taken into account. Except for research on technology that is complementary to energy (and possibly the technology for using fossil fuel), this crowding out dampens the effects of the Green Paradox.

However, the example in the discussion indicates that the results can be changed significantly, and possibly reversed if the supply of fossil fuel is driven by extraction costs rather than by the scarcity value.

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## 2.A Calculations

### 2.A.1 Calculations for model without capital

In order to calculate the expression (2.17) for the exogenous variables, start by defining $\tilde{Y}$ as

$$Y = A_Y\tilde{Y},$$

where $Y$ is given by (2.11).

Using (2.11) and (2.12) the following derivatives can be calculated:

$$\frac{\partial\tilde{Y}}{\partial(A_EY_E)} = \tilde{Y}^{\frac{1}{\sigma_Y}}\gamma_E(A_EY_E)^{-\frac{1}{\sigma_Y}} = \tilde{Y}^{\frac{1}{\sigma_Y}}\frac{G_E}{A_EY_E} \tag{2.96}$$

$$\frac{\partial\tilde{Y}}{\partial(A_LL)} = \tilde{Y}^{\frac{1}{\sigma_Y}}\gamma_L(A_LL)^{-\frac{1}{\sigma_Y}} = \tilde{Y}^{\frac{1}{\sigma_Y}}\frac{G_L}{A_LL} \tag{2.97}$$

$$\frac{\partial Y_E}{\partial(A_BB)} = Y_E^{\frac{1}{\sigma_E}}\gamma_B(A_BB)^{-\frac{1}{\sigma_E}} = Y_E^{\frac{1}{\sigma_E}}\frac{G_B}{A_BB} \tag{2.98}$$

$$\frac{\partial Y_E}{\partial(A_SS)} = Y_E^{\frac{1}{\sigma_E}}\gamma_S(A_SS)^{-\frac{1}{\sigma_E}} = Y_E^{\frac{1}{\sigma_E}}\frac{G_S}{A_SS}, \tag{2.99}$$

where the $Gs$ are defined in (2.16). Note that, for instance,

$$\frac{\partial\tilde{Y}}{\partial A_E} = \frac{\partial\tilde{Y}}{\partial(A_EY_E)}\frac{\partial A_EY_E}{\partial A_E} = \tilde{Y}^{\frac{1}{\sigma_Y}}\gamma_E(A_EY_E)^{-\frac{1}{\sigma_Y}}Y_E = \tilde{Y}^{\frac{1}{\sigma_Y}}\frac{G_E}{A_E}.$$

Using these derivatives, the marginal product of fossil fuel is

$$\begin{aligned} F_B &= A_Y\frac{\partial\tilde{Y}}{\partial B} = A_Y\frac{\partial\tilde{Y}}{\partial Y_E}\frac{\partial Y_E}{\partial B} \\ &= A_Y\tilde{Y}^{\frac{1}{\sigma_Y}}\gamma_E A_E^{\frac{\sigma_Y-1}{\sigma_Y}}Y_E^{\frac{1}{\sigma_E}-\frac{1}{\sigma_Y}}\gamma_B A_B^{\frac{\sigma_E-1}{\sigma_E}}B^{-\frac{1}{\sigma_E}}. \end{aligned}$$

The sign of expression (2.17) is the same as the sign of $F_{BX} - \theta\frac{F_BF_X}{Y}$. Thus, in order to compute this expression I need the derivatives of $Y = F$ and $F_B$ with respect to $A_Y$, $A_L$, $A_E$, $A_B$, $A_S$ and $S$. I will calculate each of these in turn.

**Derivatives with respect to $X = A_Y$:**

$$\begin{aligned} F_{A_Y} &= \tilde{Y} \\ F_{BA_Y} &= \frac{F_B}{A_Y} \end{aligned}$$

giving

$$F_{BA_Y} - \theta\frac{F_BF_{A_Y}}{Y} = \frac{F_B}{A_Y}(1-\theta).$$

**Derivatives with respect to $X = A_L$:**

$$F_{A_L} = A_Y \frac{\partial \tilde{Y}}{\partial A_L}$$
$$F_{BA_L} = F_B \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_L}$$

giving

$$\begin{aligned} F_{BA_L} - \theta \frac{F_B F_{A_L}}{Y} &= F_B \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_L} - \theta \frac{F_B A_Y}{Y} \frac{\partial \tilde{Y}}{\partial A_L} \\ &= \frac{F_B}{\tilde{Y}} \left( \frac{1}{\sigma_Y} - \theta \right) \frac{\partial \tilde{Y}}{\partial A_L}. \end{aligned}$$

**Derivatives with respect to $X = A_E$:**

$$\begin{aligned} F_{A_E} &= A_Y \frac{\partial \tilde{Y}}{\partial A_E} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \frac{G_E}{A_E} \\ F_{BA_E} &= F_B \left[ \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_E} + \frac{\sigma_Y - 1}{\sigma_Y} \frac{1}{A_E} \right] \\ &= F_B \left[ \frac{1}{\sigma_Y} A_Y \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{G_E}{A_E} + \frac{\sigma_Y}{\sigma_Y} \frac{1}{A_E} \right] \\ &= F_B \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{1}{A_E} \left[ \frac{\sigma_Y - 1}{\sigma_Y} G_L + G_E \right] \end{aligned}$$

giving

$$\begin{aligned} F_{BA_E} - \theta \frac{F_B F_{A_E}}{Y} &= F_B \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{1}{A_E} \left[ \frac{\sigma_Y - 1}{\sigma_Y} G_L + G_E \right] - \theta \frac{F_B A_Y \tilde{Y}^{\frac{1}{\sigma_Y}}}{Y} \frac{G_E}{A_E} \\ &= \frac{F_B \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}}}{A_E} \left[ \frac{\sigma_Y - 1}{\sigma_Y} G_L + (1-\theta) G_E \right]. \end{aligned}$$

**Derivatives with respect to $X = A_B$:**

$$\begin{aligned} F_{A_B} &= A_Y \frac{\partial \tilde{Y}}{\partial A_B} = A_Y \frac{\partial \tilde{Y}}{\partial Y_E} \frac{\partial Y_E}{\partial A_B} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \frac{G_E}{G_L + G_E} \frac{G_B}{A_B} \\ F_{BA_B} &= F_B \left[ \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_B} + \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{1}{Y_E} \frac{\partial Y_E}{\partial A_B} + \frac{\sigma_E - 1}{\sigma_E} \frac{1}{A_B} \right] \end{aligned}$$

giving

$$F_{BA_B} - \theta\frac{F_B F_{A_B}}{Y} =$$

$$= F_B\left[\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial A_B} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial A_B} + \frac{\sigma_E - 1}{\sigma_E}\frac{1}{A_B}\right] - \theta\frac{F_B A_Y \frac{\partial \tilde{Y}}{\partial A_B}}{Y}$$

$$= F_B\left[\left(\frac{1}{\sigma_Y} - \theta\right)\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial A_B} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial A_B} + \frac{\sigma_E - 1}{\sigma_E}\frac{1}{A_B}\right]$$

$$= F_B\left[\left(\left(\frac{1}{\sigma_Y} - \theta\right)\frac{G_E}{G_L + G_E} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial A_B} + \frac{\sigma_E - 1}{\sigma_E}\frac{1}{A_B}\right]$$

$$= \frac{F_B}{A_B}\left[\frac{\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)G_L + \left(\frac{1}{\sigma_E} - \theta\right)G_E}{G_L + G_E}\frac{G_B}{G_B + G_S} + \frac{\sigma_E - 1}{\sigma_E}\frac{G_B + G_S}{G_B + G_S}\right]$$

$$= \frac{F_B}{A_B}\frac{\left[\frac{\sigma_Y - 1}{\sigma_Y}G_L + (1-\theta)\,G_E\right]G_B + \frac{\sigma_E - 1}{\sigma_E}\left(G_L + G_E\right)G_S}{\left(G_L + G_E\right)\left(G_B + G_S\right)}.$$

**Derivatives with respect to $X = A_S$ or $X = S$:**

$$F_X = A_Y\frac{\partial \tilde{Y}}{\partial X} = A_Y\frac{\partial \tilde{Y}}{\partial Y_E}\frac{\partial Y_E}{\partial X}$$

$$F_{BX} = F_B\left[\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial X} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial X}\right]$$

giving

$$F_{BX} - \theta\frac{F_B F_X}{Y} = F_B\left[\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial X} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial X}\right] - \theta\frac{F_B}{Y}A_Y\frac{\partial \tilde{Y}}{\partial X}$$

$$= F_B\left[\left(\frac{1}{\sigma_Y} - \theta\right)\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial X} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\frac{\partial Y_E}{\partial X}\right]$$

$$= F_B\left[\left(\frac{1}{\sigma_Y} - \theta\right)\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial Y_E} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_E}\right)\frac{1}{Y_E}\right]\frac{\partial Y_E}{\partial X}$$

$$= \frac{F_B}{Y_E}\frac{\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)G_L + \left(\frac{1}{\sigma_E} - \theta\right)G_E}{G_L + G_E}\frac{\partial Y_E}{\partial X}.$$

### 2.A.2 Calculations for model with capital

I will now calculate (2.23) and (2.24) with $X$ equal to $A_Y$, $A_{KL}$, $A_E$, $A_B$, $A_S$ and $S$. These expressions contain the partial derivatives $F_B$, $F_K$,

$F_{BK}$, $F_{KK}$ $F_X$, $F_{BX}$ and $F_{KX}$. I will start by calculating the derivatives with respect to the endogenous variables. After that, I will calculate the derivatives involving each of the exogenous variables in turn. For each $X$, I then combine the derivatives into the expressions in (2.23) and (2.24).

Production is

$$Y = F(B, K; \Gamma) = A_Y \tilde{Y},$$

where

$$\tilde{Y} = \left[\gamma_{KL} \left(A_{KL} Y_{KL}\right)^{\frac{\sigma_Y - 1}{\sigma_Y}} + \gamma_E \left(A_E Y_E\right)^{\frac{\sigma_Y - 1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y - 1}},$$

with

$$Y_{KL} = K^{\alpha} L^{1-\alpha} \text{ and } Y_E = \left[\gamma_B \left(A_B B\right)^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S \left(A_S S\right)^{\frac{\sigma_E - 1}{\sigma_E}}\right]^{\frac{\sigma_E}{\sigma_E - 1}}.$$

Using the definitions of the $G$s in (2.16) and (2.25), the following derivatives can be calculated:

$$\frac{\partial \tilde{Y}}{\partial (A_F Y_F)} = \tilde{Y}^{\frac{1}{\sigma_Y}} \frac{G_F}{A_F Y_F} \text{ for } F \in \{E, KL\}$$

$$\frac{\partial Y_E}{\partial (A_F F)} = Y_E^{\frac{1}{\sigma_E}} \frac{G_F}{A_F F} \text{ for } F \in \{B, S\}$$

and

$$\frac{\partial Y_{KL}}{\partial K} = \alpha \frac{Y_{KL}}{K}$$

**Derivatives with respect to endogenous variables**

$$
\begin{aligned}
F_B &= A_Y \frac{\partial \tilde{Y}}{\partial Y_E} \frac{\partial Y_E}{\partial B} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \gamma_E A_E^{\frac{\sigma_Y - 1}{\sigma_Y}} Y_E^{\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}} \gamma_B A_B^{\frac{\sigma_E - 1}{\sigma_E}} B^{-\frac{1}{\sigma_E}} \\
&= Y \frac{G_E}{G_{KL} + G_E} \frac{G_B}{G_B + G_S} \frac{1}{B} \\
F_{BB} &= F_B \left( \frac{\frac{1}{\sigma_Y}}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial B} + \frac{\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}}{Y_E} \frac{\partial Y_E}{\partial B} - \frac{\frac{1}{\sigma_E}}{B} \right) \\
&= -F_B \left[ \frac{1}{\sigma_Y} \frac{G_{KL}}{G_E + G_{KL}} \frac{G_B}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_S}{G_B + G_S} \right] \frac{1}{B} \\
F_K &= A_Y \frac{\partial \tilde{Y}}{\partial Y_{KL}} \frac{\partial Y_{KL}}{\partial K} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \gamma_{KL}^{\frac{\sigma_Y - 1}{\sigma_Y}} Y_{KL}^{\frac{\sigma_Y - 1}{\sigma_Y}} \alpha \frac{1}{K} = Y \frac{G_{KL}}{G_{KL} + G_E} \alpha \frac{1}{K} \\
F_{KK} &= F_K \left( \frac{\frac{1}{\sigma_Y}}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial Y_{KL}} \frac{\partial Y_{KL}}{\partial K} + \frac{\frac{\sigma_Y - 1}{\sigma_Y}}{Y_{KL}} \frac{\partial Y_{KL}}{\partial K} - \frac{1}{K} \right) \\
&= -F_K \left[ 1 - \alpha + \alpha \frac{1}{\sigma_Y} \frac{G_E}{G_E + G_{KL}} \right] \frac{1}{K} \\
F_{BK} &= F_B \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial K} = F_K \frac{\frac{1}{\sigma_Y}}{\tilde{Y}} \frac{\partial d\tilde{Y}}{\partial B} = \frac{1}{\sigma_Y} \frac{F_B F_K}{Y}
\end{aligned}
$$

**Calculations of** (2.23) **and** (2.24) **for each** $X$

**Calculations for** $X = A_Y$**:**

$$
\begin{aligned}
F_{A_Y} &= \tilde{Y} \\
F_{BA_Y} &= \frac{F_B}{A_Y} = \frac{\partial \tilde{Y}}{\partial Y_E} \frac{\partial Y_E}{\partial B} \\
F_{KA_Y} &= \frac{F_K}{A_Y} = \frac{\partial \tilde{Y}}{\partial Y_{KL}} \frac{\partial Y_{KL}}{\partial K}
\end{aligned}
$$

This delivers

$$
\frac{F_{KA_Y}}{F_K} - \frac{F_{BA_Y}}{F_B} = \frac{1}{A_Y} - \frac{1}{A_Y} = 0
$$

and

$$
\begin{aligned}
&\left( \frac{F_{KK}}{F_K} Y - \theta F_K \right) \frac{F_{BA_Y}}{F_B} - \left( \frac{1}{\sigma_Y} - \theta \right) F_{KA_Y} + \theta \left( \frac{F_{BK}}{F_B} - \frac{F_{KK}}{F_K} \right) F_{A_Y} \\
&= \frac{1}{A_Y} \left[ \frac{F_{KK}}{F_K} Y - \theta F_K - \left( \frac{1}{\sigma_Y} - \theta \right) F_K + \theta \left( \frac{F_{BK}}{F_B} - \frac{F_{KK}}{F_K} \right) Y \right] \\
&= \frac{\theta - 1}{A_Y} \left[ \frac{1}{\sigma_Y} F_K - \frac{F_{KK}}{F_K} Y \right].
\end{aligned}
$$

The parenthesis in the last expression is positive, so the expression has the same sign as $\theta - 1$.

**Calculations for $X = A_{KL}$:**

$$\begin{aligned}
F_{A_{KL}} &= A_Y \frac{\partial \tilde{Y}}{\partial A_{KL}} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \frac{G_{KL}}{A_{KL}} \\
F_{BA_{KL}} &= \frac{F_B}{\sigma_Y \tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_{KL}} = \frac{F_B}{\sigma_Y} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{G_{KL}}{A_{KL}} \\
F_{KA_{KL}} &= F_K \left( \frac{1}{\sigma_Y \tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_{KL}} + \frac{\sigma_Y - 1}{\sigma_Y} \frac{1}{A_{KL}} \right) \\
&= \frac{F_K}{A_{KL}} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \left( G_{KL} + \frac{\sigma_Y - 1}{\sigma_Y} G_E \right)
\end{aligned}$$

This yields

$$\begin{aligned}
F_{BA_{KL}} - \frac{F_B}{F_K} F_{KA_{KL}} &= \frac{F_B}{\sigma_Y} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{G_{KL}}{A_{KL}} \\
&\quad - \frac{F_B}{F_K} \frac{F_K \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}}}{A_{KL}} \left( G_{KL} + \frac{\sigma_Y - 1}{\sigma_Y} G_E \right) \\
&= \frac{1 - \sigma_Y}{\sigma_Y} \frac{F_B}{A_{KL}}
\end{aligned}$$

and

$$\begin{aligned}
&\left( \frac{F_{KK}}{F_K} Y - \theta F_K \right) \frac{F_{BA_{KL}}}{F_B} - \left( \frac{1}{\sigma_Y} - \theta \right) F_{KA_{KL}} + \theta \left( \frac{F_{BK}}{F_B} - \frac{F_{KK}}{F_K} \right) F_{A_{KL}} \\
&= \left( \frac{1}{\sigma_Y} - \theta \right) \frac{\tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}}}{A_{KL}} \left[ \frac{F_{KK}}{F_K} Y G_{KL} - F_K G_{KL} - F_K \frac{\sigma_Y - 1}{\sigma_Y} G_E \right] \\
&= \left( \theta - \frac{1}{\sigma_Y} \right) \frac{\tilde{Y}^{\frac{1}{\sigma_Y}}}{A_{KL}} \frac{G_{KL}}{K} A_Y.
\end{aligned}$$

**Calculations for $X = A_E$:**

$$\begin{aligned}
F_{A_E} &= A_Y \frac{\partial \tilde{Y}}{\partial A_E} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} \frac{G_E}{A_E} \\
F_{BA_E} &= F_B \left( \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_E} + \frac{\sigma_Y - 1}{\sigma_Y} \frac{1}{A_E} \right) \\
&= \frac{F_B}{A_E} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \left( \frac{\sigma_Y - 1}{\sigma_Y} G_{KL} + G_E \right) \\
F_{KA_E} &= F_K \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_E} = \frac{F_K}{\sigma_Y} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{G_E}{A_E}
\end{aligned}$$

This leads to

$$
\begin{aligned}
F_{BA_E} - \frac{F_B}{F_K} F_{KA_E} &= \frac{F_B}{A_E} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \left( \frac{\sigma_Y - 1}{\sigma_Y} G_{KL} + G_E \right) - \frac{F_B}{\sigma_Y} \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} \frac{G_E}{A_E} \\
&= \frac{\sigma_Y - 1}{\sigma_Y} \frac{F_B}{A_E}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left( \frac{F_{KK}}{F_K} Y - \theta F_K \right) \frac{F_{BA_E}}{F_B} - \left( \frac{1}{\sigma_Y} - \theta \right) F_{KA_E} + \theta \left( \frac{F_{BK}}{F_B} - \frac{F_{KK}}{F_K} \right) F_{A_E} \\
&= \frac{\tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}}}{A_E} \begin{bmatrix} \frac{F_{KK}}{F_K} Y \left( \frac{\sigma_Y - 1}{\sigma_Y} G_{KL} + (1-\theta) G_E \right) \\ -\theta \frac{\sigma_Y - 1}{\sigma_Y} F_K \tilde{Y}^{\frac{\sigma_Y - 1}{\sigma_Y}} - \left( \frac{1}{\sigma_Y} \left( \frac{1}{\sigma_Y} - \theta \right) \right) F_K G_E \end{bmatrix} \\
&= \frac{Y \tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}}}{A_E K} \left[ \frac{1 - \sigma_Y}{\sigma_Y} \left( 1 + \alpha \left( \theta - 1 \right) \right) G_{KL} + (\theta - 1) \left( 1 + \alpha \frac{1 - \sigma_Y}{\sigma_Y} \right) G_E \right].
\end{aligned}
$$

**Calculations for $X = A_B$:**

$$
\begin{aligned}
F_{A_B} &= A_Y \frac{\partial \tilde{Y}}{\partial A_B} = A_Y \tilde{Y}^{\frac{1}{\sigma_Y}} G_E Y_E^{\frac{1-\sigma_E}{\sigma_E}} \frac{G_B}{A_B} \\
F_{BA_B} &= F_B \left[ \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_B} + \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{1}{Y_E} \frac{\partial Y_E}{\partial A_B} + \frac{\sigma_E - 1}{\sigma_E} \frac{1}{A_B} \right] \\
&= F_B \left[ -\frac{1}{\sigma_Y} \frac{G_{KL}}{G_E + G_{KL}} \frac{G_B}{G_B + G_S} + \left( 1 - \frac{1}{\sigma_E} \frac{G_S}{G_B + G_S} \right) \right] \frac{1}{A_B} \\
F_{KA_B} &= F_K \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_B} = F_K \frac{1}{\sigma_Y} \frac{\tilde{Y}^{\frac{1-\sigma_Y}{\sigma_Y}} Y_E^{\frac{1-\sigma_E}{\sigma_E}}}{A_S} G_E G_B
\end{aligned}
$$

This delivers

$$
\begin{aligned}
F_{BA_B} - \frac{F_B}{F_K} F_{KA_B} &= F_B \left[ \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_B} + \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{1}{Y_E} \frac{\partial Y_E}{\partial A_B} \right] \\
&\quad + F_B \frac{\sigma_E - 1}{\sigma_E} \frac{1}{A_B} - \frac{F_B}{F_K} F_K \frac{1}{\sigma_Y} \frac{1}{\tilde{Y}} \frac{\partial \tilde{Y}}{\partial A_B} \\
&= F_B \frac{Y_E^{\frac{1-\sigma_E}{\sigma_E}}}{A_B} \left[ \frac{\sigma_Y - 1}{\sigma_Y} G_B + \frac{\sigma_E - 1}{\sigma_E} G_S \right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{F_{KK}}{F_K}Y - \theta F_K\right)\frac{F_{BA_B}}{F_B} - \left(\frac{1}{\sigma_Y} - \theta\right)F_{KA_B} + \theta\left(\frac{F_{BK}}{F_B} - \frac{F_{KK}}{F_K}\right)F_{A_B} \\
&= \frac{F_{KK}}{F_K}Y\left(\left(\frac{1}{\sigma_Y} - \theta\right)\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial A_B} + \frac{Y_E^{\frac{1-\sigma_E}{\sigma_E}}}{A_B}\left(\frac{\sigma_Y - 1}{\sigma_Y}G_B + \frac{\sigma_E - 1}{\sigma_E}G_S\right)\right) \\
&\quad - F_K\left(\left(\frac{1}{\sigma_Y} - \theta\right)\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial A_B} + \theta\frac{Y_E^{\frac{1-\sigma_E}{\sigma_E}}}{A_B}\left(\frac{\sigma_Y - 1}{\sigma_Y}G_B + \frac{\sigma_E - 1}{\sigma_E}G_S\right)\right) \\
&= -\left[\begin{array}{l}
\left(\begin{array}{c}(\theta-1)\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)\frac{G_E}{G_{KL}+G_E} \\ +\frac{\sigma_Y-1}{\sigma_Y}\left(1+\alpha(\theta-1)\right)\frac{G_{KL}}{G_{KL}+G_E}\end{array}\right)\frac{G_B}{G_B+G_S} \\
+\left(\begin{array}{c}\left(1+\alpha(\theta-1)\right)\frac{G_{KL}}{G_{KL}+G_E} \\ +\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)\frac{G_E}{G_{KL}+G_E}\end{array}\right)\frac{\sigma_E-1}{\sigma_E}\frac{G_S}{G_B+G_S}
\end{array}\right]\frac{Y}{K}\frac{1}{A_B}.
\end{aligned}
$$

**Calculations for $X \in \{A_S, S\}$:**

$$
\begin{aligned}
F_X &= A_Y\frac{\partial \tilde{Y}}{\partial X} = A_Y\tilde{Y}^{\frac{1}{\sigma_Y}}G_E Y_E^{\frac{1-\sigma_E}{\sigma_E}}\frac{G_S}{X} \\
F_{BX} &= F_B\left[\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial Y_E} + \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{Y_E}\right]\frac{\partial Y_E}{\partial X} \\
&= F_B\left[\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_{KL}}{G_{KL}+G_E}\right]\frac{G_S}{G_B+G_S}\frac{1}{X} \\
F_{KX} &= F_K\frac{1}{\sigma_Y}\frac{1}{\tilde{Y}}\frac{\partial \tilde{Y}}{\partial X} = F_K\frac{1}{\sigma_Y}\frac{G_E}{G_{KL}+G_E}\frac{G_S}{G_B+G_S}\frac{1}{X}
\end{aligned}
$$

This gives

$$
\begin{aligned}
F_{BX} - \frac{F_B}{F_K}F_{KX} &= F_B\left[\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_{KL}}{G_{KL}+G_E}\right]\frac{G_S}{G_B+G_S}\frac{1}{X} \\
&\quad - \frac{F_B}{F_K}F_K\frac{1}{\sigma_Y}\frac{G_E}{G_{KL}+G_E}\frac{G_S}{G_B+G_S}\frac{1}{X} \\
&= \left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{F_B}{X}\frac{G_S}{G_B+G_S}
\end{aligned}
$$

and

$$\left(\frac{F_{KK}}{F_K}Y-\theta F_K\right)\frac{F_{BX}}{F_B}-\left(\frac{1}{\sigma_Y}-\theta\right)F_{KX}+\theta\left(\frac{F_{BK}}{F_B}-\frac{F_{KK}}{F_K}\right)F_X$$
$$=\left[\begin{array}{c}\frac{F_{KK}}{F_K}Y\left(\frac{1}{\sigma_E}-\frac{1}{\sigma_Y}\frac{G_{KL}}{G_{KL}+G_E}-\theta\frac{G_E}{G_{KL}+G_E}\right)\\+\left(\theta\left(\frac{1}{\sigma_Y}-\frac{1}{\sigma_E}\right)+\frac{1}{\sigma_Y}\left(\theta-\frac{1}{\sigma_Y}\right)\frac{G_E}{G_{KL}+G_E}\right)F_K\end{array}\right]\frac{G_S}{G_B+G_S}\frac{1}{X}$$
$$=\left[\begin{array}{c}\left(\theta-\frac{1}{\sigma_E}\right)\left(1+\alpha\frac{1-\sigma_Y}{\sigma_Y}\right)\frac{G_E}{G_{KL}+G_E}\\+\left(\frac{1}{\sigma_Y}-\frac{1}{\sigma_E}\right)\left(1+\alpha\left(\theta-1\right)\right)\frac{G_{KL}}{G_{KL}+G_E}\end{array}\right]\frac{G_S}{G_B+G_S}\frac{Y}{K}\frac{1}{X}.$$

### 2.A.3 Derivation of equation (2.71)

Substituting (2.70) into (2.69) gives

$$\begin{aligned}&\beta^{t_2}H_{1,t_2}\left[(1-\theta)\frac{D'_{t_2}}{D_{t_2}}\left(M'_{t_1}+B'_{t_1}\right)-\xi_{B,t_2}B'_{t_2}\right]\\&-\beta^{t_1}H_{1,t_1}\left[(1-\theta)\frac{D'_{t_1}}{D_{t_1}}M'_{t_1}-\xi_{B,t_1}B'_{t_1}\right]\\=&H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}B'_{t_2}\\&+\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\frac{D'_t}{D_t}+(1-\theta)\frac{D'_t}{D_t}\right)\left(M'_{t_1}+B'_{t_1}\right).\end{aligned}$$

Solving for $B'_2$ then leads to

$$\begin{aligned}B'_{t_2}=&\frac{\beta^{t_2}H_{1,t_2}(1-\theta)\frac{D'_{t_2}}{D_{t_2}}+\beta^{t_1}H_{1,t_1}\xi_{B,t_1}}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}B'_{t_1}\\&-\frac{\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\frac{D'_t}{D_t}+(1-\theta)\frac{D'_t}{D_t}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}B'_{t_1}\\&+\frac{\beta^{t_2}H_{1,t_2}(1-\theta)\frac{D'_{t_2}}{D_{t_2}}-\beta^{t_1}H_{1,t_1}(1-\theta)\frac{D'_{t_1}}{D_{t_1}}}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1}\\&-\frac{\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\frac{D'_t}{D_t}+(1-\theta)\frac{D'_t}{D_t}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1}.\end{aligned}$$

Adding and subtracting $\beta^{t_2}H_{1,t_2}(1-\theta)\frac{D'_{t_1}}{D_{t_1}}$ and $\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\frac{D'_{t_1}}{D_{t_1}}$ in the numerator implies

$$
\begin{aligned}
B'_{t_2} = & \frac{\beta^{t_2}H_{1,t_2}(1-\theta)\frac{D'_{t_2}}{D_{t_2}}+\beta^{t_1}H_{1,t_1}\xi_{B,t_1}}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}B'_{t_1} \\
& -\frac{\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\theta\frac{D'_t}{D_t}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}B'_{t_1} \\
& +\frac{\beta^{t_2}H_{1,t_2}(1-\theta)\left(\frac{D'_{t_2}}{D_{t_2}}-\frac{D'_{t_1}}{D_{t_1}}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1} \\
& -\frac{\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\frac{D_{t'}}{D_t}+(1-\theta)\left(\frac{D'_t}{D_t}-\frac{D'_{t_1}}{D_{t_1}}\right)\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1} \\
& +\frac{(1-\theta)\left(\beta^{t_2}H_{2,t_2}-\beta^{t_1}H_{1,t_1}-\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\right)\frac{D'_{t_1}}{D_{t_1}}}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1}.
\end{aligned}
$$

Given that the carbon cycle fulfills (2.32), the first-order condition of the planner problem (use (2.34) in (2.58)) implies that the last term is zero. This produces

$$
\begin{aligned}
B'_{t_2} = & \frac{\beta^{t_2}H_{1,t_2}(1-\theta)\frac{D'_{t_2}}{D_{t_2}}+\beta^{t_1}H_{1,t_1}\xi_{B,t_1}-\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\theta\frac{D'_t}{D_t}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}B'_{t_1} \\
& +\frac{\beta^{t_2}H_{1,t_2}(1-\theta)\left(\frac{D'_{t_2}}{D_{t_2}}-\frac{D'_{t_1}}{D_{t_1}}\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1} \\
& -\frac{\sum_{t=t_1+1}^{t_2}H_{2,t_1,t}\left(\frac{D''_t}{D'_t}-\frac{D_{t'}}{D_t}+(\theta-1)\left(\frac{D'_{t_1}}{D_{t_1}}-\frac{D'_t}{D_t}\right)\right)}{\beta^{t_2}H_{1,t_2}+H_{2,t_1,t_2}(1-\theta)\frac{F_{B,t_2}}{F_{t_2}}}M'_{t_1}.
\end{aligned}
$$

### 2.A.4 Calculations for model with elastic supply of alternative-energy input

I will now calculate the effects of varying the exogenous variables when the supply of alternative energy is endogenous. Throughout, the $G$s are defined as in (2.16). I will start by calculating the derivatives of the production function with respect to the endogenous variables and computing some combinations of derivatives that are useful in the subsequent calculations. I will then calculate the derivative of $\tilde{H}$ with respect to $B$.

After that, I will calculate the derivative of $\tilde{H}$ with respect to each of the exogenous variables.

**Derivatives with respect to endogenous variables**

Using the fact that $Y = A_Y \tilde{Y}$ and the derivatives (2.96)-(2.99) from 2.A.1, the marginal products, in final goods production, of labor and the alternative-energy input, respectively, are

$$
\begin{aligned}
F_L &= F \frac{G_L}{G_E + G_L} \frac{1}{L^Y} \\
F_S &= F \frac{G_E}{G_E + G_L} \frac{G_S}{G_B + G_S} \frac{1}{S}.
\end{aligned}
$$

The equilibrium labor allocation condition (2.84) then gives

$$
\frac{G_L}{G_E + G_L} \frac{1}{L^Y} = \frac{G_E}{G_E + G_L} \frac{G_S}{G_B + G_S} \frac{a_S}{S}. \tag{2.100}
$$

A common term for all derivatives is $2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}$. The derivatives needed to calculate this expression are

$$
\begin{aligned}
F_{LS} &= F_L \frac{1}{\sigma_Y} \frac{G_E}{G_E + G_L} \frac{G_S}{G_B + G_S} \frac{1}{S} \\
F_{LL} &= -F_L \frac{1}{\sigma_Y} \frac{G_E}{G_E + G_L} \frac{1}{L^Y} \\
F_{SS} &= -F_S \left[ \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B + G_S} \right] \frac{1}{S}.
\end{aligned}
$$

Furthermore, the derivatives involving fossil-fuel use are

$$
\begin{aligned}
F_B &= F \frac{G_E}{G_E + G_L} \frac{G_B}{G_B + G_S} \frac{1}{B} \\
F_{BB} &= -F_B \left[ \frac{1}{\sigma_E} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} \frac{G_B}{G_B + G_S} \right] \frac{1}{B} \\
F_{BS} &= F_B \left[ \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \frac{G_L}{G_L + G_E} \right] \frac{G_S}{G_B + G_S} \frac{1}{S} \\
&= F_S \left[ \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \frac{G_L}{G_L + G_E} \right] \frac{G_B}{G_B + G_S} \frac{1}{B} \\
F_{BL} &= F_B \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} \frac{1}{L^Y} \\
&= F_L \frac{1}{\sigma_Y} \frac{G_E}{G_E + G_L} \frac{G_B}{G_B + G_S} \frac{1}{B}.
\end{aligned}
$$

Combining these derivatives and using (2.100) we obtain

$$
\begin{aligned}
F_{BB} - \theta \frac{F_B^2}{F} &= -F_B \frac{1}{\sigma_E} \frac{G_S}{G_B + G_S} \frac{1}{B} \\
&\quad - F_B \left( \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} + \theta \frac{G_E}{G_E + G_L} \right) \frac{G_B}{G_B + G_S} \frac{1}{B} \\
a_S F_{BS} - F_{BL} &= F_L \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{G_B}{G_B + G_S} \frac{1}{B} \\
&= F_B \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{G_S}{G_B + G_S} \frac{a_S}{S}
\end{aligned}
$$

$$
\begin{aligned}
2a_S F_{LS} - F_{LL} - a_S^2 F_{SS} &= F_L \left( \frac{1}{\sigma_Y} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B + G_S} \right) \frac{a_S}{S} \\
&\quad + F_L \frac{1}{\sigma_Y} \frac{1}{L^Y}. \qquad (2.101)
\end{aligned}
$$

**Calculation of $\frac{\partial \tilde{H}}{\partial B}$**

From (2.88), we see that

$$
\frac{1}{u'(C)} \frac{\partial \tilde{H}}{\partial B} = F_{BB} - \theta \frac{F_B^2}{F} + \frac{(F_{BS} a_S - F_{BL})^2}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.
$$

Using the derivatives and expressions from the previous section, one arrives at

$$
\begin{aligned}
\frac{1}{u'(C)} \frac{\partial \tilde{H}}{\partial B} = &- \frac{F_B}{B} \frac{\frac{1}{\sigma_Y \sigma_E} \left( 1 - \frac{G_E}{G_E + G_L} \left( \frac{G_B}{G_B + G_S} \right)^2 \right) \frac{a_S}{S}}{\left( \frac{1}{\sigma_Y} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B + G_S} \right) \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L^Y}} \\
&- \frac{F_B}{B} \frac{\frac{\theta}{\sigma_E} \frac{G_E}{G_E + G_L} \left( \frac{G_B}{G_B + G_S} \right)^2 \frac{a_S}{S}}{\left( \frac{1}{\sigma_Y} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B + G_S} \right) \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L^Y}} \qquad (2.102) \\
&- \frac{F_B}{B} \frac{\frac{1}{\sigma_Y} \left( \frac{1}{\sigma_E} \frac{G_S}{G_B + G_S} + \theta \frac{G_B}{G_B + G_S} \right) \frac{1}{L^Y}}{\left( \frac{1}{\sigma_Y} \frac{G_S}{G_B + G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B + G_S} \right) \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L^Y}}.
\end{aligned}
$$

This expression is negative.

**Calculation of $\frac{\partial \tilde{H}}{\partial A_Y}$**

From (2.91), we have that

$$
\frac{1}{u'(C)} \frac{\partial \tilde{H}}{\partial A_Y} = F_{BA_Y} - \theta \frac{F_B F_{A_Y}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_Y} - F_{LA_Y})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.
$$

The derivatives of the production function with respect to $A_Y$ are

$$\begin{aligned}
F_{A_Y} &= F\frac{1}{A_Y} \\
F_{BA_Y} &= F_B\frac{1}{A_Y} \\
F_{SA_Y} &= F_S\frac{1}{A_Y} \\
F_{LA_Y} &= F_L\frac{1}{A_Y}.
\end{aligned}$$

Combining the last two derivatives delivers

$$a_S F_{SA_Y} - F_{LA_Y} = (a_S F_S - F_L)\,\frac{1}{A_Y} = \{(2.84)\} = 0$$

This implies that the reallocation of labor satisfies

$$L^S_{A_Y} = \frac{a_S F_{SA_Y} - F_{LA_Y}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} = 0$$

The first two derivatives then yield

$$\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_Y} = F_{BA_Y} - \theta\frac{F_B F_{A_Y}}{F} = (1-\theta)F_B\frac{1}{A_Y}$$

**Calculation of $\frac{\partial \tilde{H}}{\partial A_L}$**

From (2.91), we obtain

$$\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_L} = F_{BA_L} - \theta\frac{F_B F_{A_L}}{F} + \frac{(a_S F_{BS} - F_{BL})\,(a_S F_{SA_L} - F_{LA_L})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.$$

The derivatives of the production function with respect to $A_L$ are

$$\begin{aligned}
F_{A_L} &= F\frac{G_L}{G_E+G_L}\frac{1}{A_L} \\
F_{BA_L} &= F_B\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{1}{A_L} \\
F_{SA_L} &= F_S\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{1}{A_L} \\
F_{LA_L} &= F_L\left(1-\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\right)\frac{1}{A_L}.
\end{aligned}$$

Combining the first two derivatives yields

$$\begin{aligned} F_{BA_L} - \theta \frac{F_B F_{A_L}}{F} &= F_B \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} \frac{1}{A_L} - \theta F_B \frac{G_L}{G_E + G_L} \frac{1}{A_L} \\ &= F_B \left( \frac{1}{\sigma_Y} - \theta \right) \frac{G_L}{G_E + G_L} \frac{1}{A_L}. \end{aligned}$$

Combining the last two derivatives delivers

$$\begin{aligned} a_S F_{SA_L} - F_{LA_L} &= a_S F_S \frac{1}{\sigma_Y} \frac{G_L}{G_E + G_L} \frac{1}{A_L} - F_L \left( 1 - \frac{1}{\sigma_Y} \frac{G_E}{G_E + G_L} \right) \frac{1}{A_L} \\ &= F_L \frac{1 - \sigma_Y}{\sigma_Y} \frac{1}{A_L}. \end{aligned}$$

This delivers a reallocation of labor satisfying

$$L^S_{A_L} = \frac{a_S F_{SA_L} - F_{LA_L}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} = \frac{F_L \frac{1-\sigma_Y}{\sigma_Y} \frac{1}{A_L}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}},$$

which is positive under assumption (2.13).

The change in the marginal value of fossil-fuel use is

$$\begin{aligned} \frac{1}{u'(C)} \frac{\partial \tilde{H}}{\partial A_L} &= F_{BA_L} - \theta \frac{F_B F_{A_L}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_L} - F_{LA_L})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} \\ &= \frac{F_B \left( \frac{1}{\sigma_Y} - \theta \right) \frac{G_L}{G_E+G_L} \frac{1}{A_L} F_L \left( \frac{\frac{1}{\sigma_Y} G_S + \frac{1}{\sigma_E} G_B}{G_B+G_S} \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L} \right)}{F_L \left( \frac{1}{\sigma_Y} \frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B+G_S} \right) \frac{a_S}{S} + F_L \frac{1}{\sigma_Y} \frac{1}{L^Y}} \\ &\quad + \frac{F_B \left( \frac{1}{\sigma_E} - \frac{1}{\sigma_Y} \right) \frac{G_S}{G_B+G_S} \frac{a_S}{S} F_L \frac{1-\sigma_Y}{\sigma_Y} \frac{1}{A_L}}{F_L \left( \frac{1}{\sigma_Y} \frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B+G_S} \right) \frac{a_S}{S} + F_L \frac{1}{\sigma_Y} \frac{1}{L^Y}} \\ &= \frac{F_B}{A_L} \frac{\frac{1}{\sigma_Y} \left( 1 + \frac{1-\sigma_Y}{\sigma_E} - \theta \right) \frac{G_S}{G_B+G_S}}{\left( \frac{1}{\sigma_Y} \frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B+G_S} \right) \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L^Y}} \frac{a_S}{S}. \\ &\quad + \frac{F_B}{A_L} \frac{\left( \frac{1}{\sigma_Y} - \theta \right) \frac{1}{\sigma_E} \frac{G_L}{G_E+G_L} \frac{G_B}{G_B+G_S}}{\left( \frac{1}{\sigma_Y} \frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E} \frac{G_B}{G_B+G_S} \right) \frac{a_S}{S} + \frac{1}{\sigma_Y} \frac{1}{L^Y}} \frac{a_S}{S}. \end{aligned}$$

In general, the sign of this is expression is ambiguous. If $\theta \leq 1 + \frac{1-\sigma_Y}{\sigma_E} > 1$ it is unambiguously positive. If $\theta > 1 + \frac{1-\sigma_Y}{\sigma_E}$, the first term is negative while the second term is positive. When the supply of the alternative-energy input was inelastic, the marginal value of fossil-fuel use was increasing in $A_L$ under assumptions (2.13), (2.14) and (2.15).

Since $a_S F_{SA_L} - F_{LA_L} = F_L \frac{1-\sigma_Y}{\sigma_Y}\frac{1}{A_L} > 0$, it follows from (2.89) that labor is moved from final good production to the production of the alternative energy input. This reallocation counteracts the positive effect on the value of fossil-fuel use and leaves us with an ambiguity.

**Calculation of $\frac{\partial \tilde{H}}{\partial A_E}$**

From (2.91) we arrive at

$$\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_E} = F_{BA_E} - \theta\frac{F_B F_{A_E}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_E} - F_{LA_E})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.$$

The derivatives of the production function with respect to $A_E$ are

$$\begin{aligned} F_{A_E} &= F\frac{G_E}{G_E+G_L}\frac{1}{A_E} \\ F_{BA_E} &= F_B\left(1-\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{1}{A_E} \\ F_{SA_E} &= F_S\left(1-\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{1}{A_E} \\ F_{LA_E} &= F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\frac{1}{A_E}. \end{aligned}$$

Combining the first two derivatives delivers

$$\begin{aligned} F_{BA_E} - \theta\frac{F_B F_{A_E}}{F} &= F_B\left(1-\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{1}{A_E} - \theta F_B\frac{G_E}{G_E+G_L}\frac{1}{A_E} \\ &= F_B\left(1-\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L} - \theta\frac{G_E}{G_E+G_L}\right)\frac{1}{A_E}. \end{aligned}$$

Combining the last two derivatives gives

$$\begin{aligned} a_S F_{SA_E} - F_{LA_E} &= a_S F_S\left(1-\frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{1}{A_E} - F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\frac{1}{A_E} \\ &= F_L\frac{\sigma_Y-1}{\sigma_Y}\frac{1}{A_E}. \end{aligned}$$

This produces a reallocation of labor of

$$L^S_{A_E} = \frac{a_S F_{SA_E} - F_{LA_E}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} = \frac{F_L\frac{\sigma_Y-1}{\sigma_Y}\frac{1}{A_E}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}},$$

which is negative under assumption (2.13).

The change in the marginal value of fossil-fuel use is

$$
\begin{aligned}
\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_E} &= F_{BA_E} - \theta\frac{F_B F_{A_E}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_E} - F_{LA_E})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} \\
&= F_B\left(1 - \frac{1}{\sigma_Y}\frac{G_L}{G_E + G_L} - \theta\frac{G_E}{G_E + G_L}\right)\frac{1}{A_E} \\
&\quad + \frac{F_B\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{G_S}{G_B+G_S}\frac{a_S}{S}F_L\frac{\sigma_Y - 1}{\sigma_Y}\frac{1}{A_E}}{F_L\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + F_L\frac{1}{\sigma_Y}\frac{1}{L^Y}} \\
&= -\frac{F_B}{A_E}\frac{\frac{1}{\sigma_E}\left[\frac{1-\sigma_Y}{\sigma_Y}\left(\frac{G_L}{G_E+G_L}\frac{G_B}{G_B+G_S} + \frac{G_S}{G_B+G_S}\right)\right]\frac{a_S}{S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}} \\
&\quad - \frac{F_B}{A_E}\frac{\frac{\theta-1}{\sigma_E}\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{a_S}{S} + \frac{\theta-1}{\sigma_Y}\frac{1}{L^Y}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}.
\end{aligned}
$$

This expression is negative under assumptions (2.13), (2.14) and (2.15).

**Calculation of $\frac{\partial \tilde{H}}{\partial A_B}$**

From (2.91) we see that

$$
\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_B} = F_{BA_B} - \theta\frac{F_B F_{A_B}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_B} - F_{LA_B})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.
$$

The derivatives of the production function with respect to $A_B$ are

$$
\begin{aligned}
F_{A_B} &= F\frac{G_E}{G_E + G_L}\frac{G_B}{G_B + G_S}\frac{1}{A_B} \\
F_{BA_B} &= F_B\left(1 - \frac{1}{\sigma_E}\frac{G_S}{G_B + G_S} - \frac{1}{\sigma_Y}\frac{G_L}{G_E + G_L}\frac{G_B}{G_B + G_S}\right)\frac{1}{A_B} \\
F_{SA_B} &= F_S\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E + G_L}\right)\frac{G_B}{G_B + G_S}\frac{1}{A_B} \\
F_{LA_B} &= F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E + G_L}\frac{G_B}{G_B + G_S}\frac{1}{A_B}.
\end{aligned}
$$

Combining the first two derivatives gives

$$\begin{aligned}
F_{BA_B} - \theta\frac{F_B F_{A_B}}{F} &= F_B\left(1 - \frac{1}{\sigma_E}\frac{G_S}{G_B+G_S} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{G_B}{G_B+G_S}\right)\frac{1}{A_B} \\
&\quad -\theta F_B\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{1}{A_B} \\
&= F_B\frac{\sigma_E-1}{\sigma_E}\frac{G_S}{G_B+G_S}\frac{1}{A_B} \\
&\quad -F_B\frac{1-\sigma_Y}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{1}{A_B} \\
&\quad -(\theta-1)\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{1}{A_B}.
\end{aligned}$$

Combining the last two derivatives delivers

$$\begin{aligned}
a_S F_{SA_B} - F_{LA_B} &= a_S F_S\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{G_B}{G_B+G_S}\frac{1}{A_B} \\
&\quad -F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{1}{A_B} \\
&= F_L\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{G_B}{G_B+G_S}\frac{1}{A_B}.
\end{aligned}$$

The reallocation of labor then becomes

$$L^S_{A_B} = \frac{a_S F_{SA_E} - F_{LA_E}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} = \frac{F_L\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{G_B}{G_B+G_S}\frac{1}{A_B}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}},$$

which is negative under assumption (2.13).

The change in the marginal value of fossil-fuel use is

$$\begin{aligned}
\frac{1}{u'(C)}\frac{\partial\tilde{H}}{\partial A_B} &= F_{BA_B} - \theta\frac{F_B F_{A_B}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_B} - F_{LA_B})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} \\
&= \frac{F_B}{A_B}\frac{\sigma_E-1}{\sigma_E}\frac{G_S}{G_B+G_S} \\
&\quad -\frac{F_B}{A_B}\left(\frac{1-\sigma_Y}{\sigma_Y}\frac{G_L}{G_E+G_L} + (\theta-1)\frac{G_E}{G_E+G_L}\right)\frac{G_B}{G_B+G_S} \\
&\quad \frac{F_B}{A_B}\frac{\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)^2\frac{G_B}{G_B+G_S}\frac{G_S}{G_B+G_S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{a_S}{S}.
\end{aligned}$$

The first two terms give the same behavior as in the model with inelastic supply of the alternative-energy input. The fourth term is positive. It captures the fact that an increase in $A_B$ reallocates labor from provision of the alternative-energy input to final goods production. Note that when either $G_B << G_S$ or $G_S << G_B$, the fourth term is small and the sign of $\frac{\partial \tilde{H}}{\partial A_B}$ is the same as in the model with inelastic supply of the alternative-energy input.

**Calculation of $\frac{\partial \tilde{H}}{\partial A_S}$**

From (2.91), we note that

$$\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_S} = F_{BA_S} - \theta\frac{F_B F_{A_S}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_S} - F_{LA_S})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}}.$$

The derivatives of the production function with respect to $A_S$ are

$$\begin{aligned}
F_{A_S} &= F\frac{G_E}{G_E+G_L}\frac{G_S}{G_B+G_S}\frac{1}{A_S}\\
F_{BA_S} &= F_B\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{G_S}{G_B+G_S}\frac{1}{A_S}\\
F_{SA_S} &= F_S\left(1 - \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{G_S}{G_B+G_S}\right)\frac{1}{A_S}\\
F_{LA_S} &= F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\frac{G_S}{G_B+G_S}\frac{1}{A_S}.
\end{aligned}$$

Combining the first two derivatives produces

$$\begin{aligned}
F_{BA_S} - \theta\frac{F_B F_{A_S}}{F} &= F_B\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\right)\frac{G_S}{G_B+G_S}\frac{1}{A_S}\\
&\quad -\theta F_B\frac{G_E}{G_E+G_L}\frac{G_S}{G_B+G_S}\frac{1}{A_S}\\
&= F_B\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L} - \theta\frac{G_E}{G_E+G_L}\right)\frac{G_S}{G_B+G_S}\frac{1}{A_S}.
\end{aligned}$$

Combining the last two derivatives yields

$$\begin{aligned}
a_S F_{SA_S} - F_{LA_S} &= a_S F_S\left(1 - \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L}\frac{G_S}{G_B+G_S}\right)\frac{1}{A_S}\\
&\quad -F_L\frac{1}{\sigma_Y}\frac{G_E}{G_E+G_L}\frac{G_S}{G_B+G_S}\frac{1}{A_S}\\
&= F_L\left(\frac{\sigma_E-1}{\sigma_E}\frac{G_B}{G_B+G_S} + \frac{\sigma_Y-1}{\sigma_Y}\frac{G_S}{G_B+G_S}\right)\frac{1}{A_S}.
\end{aligned}$$

This implies that the reallocation of labor becomes

$$L^S_{A_S} = \frac{a_S F_{SA_E} - F_{LA_E}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} = \frac{F_L\left(\frac{\sigma_E-1}{\sigma_E}\frac{G_B}{G_B+G_S} + \frac{\sigma_Y-1}{\sigma_Y}\frac{G_S}{G_B+G_S}\right)\frac{1}{A_S}}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}},$$

the sign of which is ambiguous under assumption (2.13). It is positive if $G_B > \frac{\sigma_E}{\sigma_E-1}\frac{\sigma_Y-1}{\sigma_Y}G_S$ and negative if $G_B < \frac{\sigma_E}{\sigma_E-1}\frac{\sigma_Y-1}{\sigma_Y}G_S$.

The change in the marginal value of fossil-fuel use is

$$\begin{aligned}\frac{1}{u'(C)}\frac{\partial \tilde{H}}{\partial A_S} &= F_{BA_S} - \theta\frac{F_S F_{A_S}}{F} + \frac{(a_S F_{BS} - F_{BL})(a_S F_{SA_S} - F_{LA_S})}{2a_S F_{LS} - F_{LL} - a_S^2 F_{SS}} \\ &= -\frac{F_B}{A_S}\frac{G_S}{G_B+G_S}\frac{\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\left(1 - \frac{1}{\sigma_E}\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}} \\ &\quad -\frac{F_B}{A_S}\frac{G_S}{G_B+G_S}\frac{\frac{1}{\sigma_E}\left(\theta - \frac{1}{\sigma_E}\right)\frac{G_E}{G_E+G_L}\frac{G_B}{G_B+G_S}\frac{a_S}{S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}} \\ &\quad -\frac{F_B}{A_S}\frac{G_S}{G_B+G_S}\frac{\frac{1}{\sigma_Y}\left(\theta - \frac{1}{\sigma_E}\right)\frac{1}{L^Y}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}\end{aligned}$$

This is negative under assumptions (2.13) and(2.14).

### Calculation of $\frac{\partial \tilde{H}}{\partial a_S}$

Using the expressions for $F_{LS}$ and $F_{SS}$ from 2.A.4 and the labor allocation condition (2.84) we obtain

$$F_S + a_S L^S F_{SS} - L^S F_{LS} = F_S\left(\frac{\sigma_Y - 1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{\sigma_E - 1}{\sigma_E}\frac{G_B}{G_B+G_S}\right).$$

From (2.93), the reallocation of labor becomes

$$L^S_{a_S} = \frac{F_S + a_S L^S F_{SS} - L^S F_{LS}}{2a_S F_{LS} - a_S^2 F_{SS} - F_{LL}} = F_S\frac{\frac{\sigma_Y-1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{\sigma_E-1}{\sigma_E}\frac{G_B}{G_B+G_S}}{2a_S F_{LS} - a_S^2 F_{SS} - F_{LL}}.$$

The sign of this expression is ambiguous under assumptions (2.13).

$$F_{BS} - \theta\frac{F_B F_S}{F} = F_B\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\frac{G_L}{G_E+G_L} - \theta\frac{G_E}{G_E+G_L}\right)\frac{1}{S}.$$

Using (2.94), we obtain

$$
\begin{aligned}
\frac{1}{u'(F)}\frac{\partial \tilde{H}}{\partial a_S} &= \left(F_{BS} - \theta\frac{F_B F_S}{F}\right) L^S \\
&\quad + \frac{\left(F_S + a_S L^S F_S S - L^S F_{LS}\right)\left(a_S F_{BS} - F_{BL}\right)}{2a_S F_{LS} - a_S^2 F_{SS} - F_{LL}} \\
&= -\frac{F_B}{a_S}\frac{\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\left(1 - \frac{1}{\sigma_E}\frac{G_E}{G_L+G_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{G_S}{G_B+G_S} \\
&\quad - \frac{F_B}{a_S}\frac{\frac{1}{\sigma_E}\left(\theta - \frac{1}{\sigma_E}\right)\frac{G_E}{G_L+G_E}\frac{G_B}{G_B+G_S}\frac{a_S}{S}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{G_S}{G_B+G_S} \\
&\quad - \frac{F_B}{a_S}\frac{\frac{1}{\sigma_Y}\left(\theta - \frac{1}{\sigma_E}\right)\frac{1}{L^Y}}{\left(\frac{1}{\sigma_Y}\frac{G_S}{G_B+G_S} + \frac{1}{\sigma_E}\frac{G_B}{G_B+G_S}\right)\frac{a_S}{S} + \frac{1}{\sigma_Y}\frac{1}{L^Y}}\frac{G_S}{G_B+G_S}.
\end{aligned}
$$

This is negative under assumptions (2.13)-(2.15).

# Chapter 3

# The Role of the Nature of Damages

## 3.1 Introduction

The emissions of $CO_2$ from the burning of fossil fuels is believed to be an important driver behind climate change. Increasing concentrations of greenhouse gases in the atmosphere strengthens the green house effect and thereby increases the temperature on earth. With higher temperature follow many changes in the functioning of the earth system. The IPCC assessment report (2007) on climate change describes a large range of effects of climate change. The discussed effects include a sea level rise, heat waves, storms, changes in disease vectors, agricultural productivities and water availability.

From these examples of the expected effects, it can be seen that climate change will affect the economy in many different ways. When building integrated assessment models, with feedbacks between the climate and the economy, the effects of a changing climate on the economy must be modeled explicitly and a decision must be made about exactly how the economy is affected. In this chapter I will investigate the consequences of different possible choices.

Following Nordhaus' groundbreaking work, with the DICE/RICE models (see, e.g., Nordhaus and Boyer, 2000), the most common way of modeling these effects is to assume that they affect productivity. That is, that climate change affects the economy by changing - typically reducing - general TFP.

Other authors assume that climate variables instead enter directly into the utility function. Examples of this includes Acemoglu et al. 2012 and van der Ploeg and Withagen (2012).

Weitzman (2010) investigates how the shape of the damage function affects the results derived, concerning, e.g., optimal policy. He compares the effects of assuming that the climate state interacts multiplicatively or additively with production capacity to provide welfare. He finds that this choice is very important, especially if a calibrated model is extrapolated to situations with significant climate change. While the analysis is carried out using a utility function, the results also apply to a situation where damages enter into the production function.

Sterner and Persson (2008) argue that climate affects non-market goods and services. This can significantly increase the estimates of the costs related to future climate change induced damages if it is difficult to substitute the non-market goods and services with manufactured goods. Here the damages enter the production function, but not as the commonly assumed multiplicative term. Their work can be seen as an illustration of the importance of the choice described by Weitzman.

In this chapter I will consider a different alternative for how climate change can affect the economy: through increased depreciation of capital. Examples of effects that should affect depreciation of capital include flooding and storms. As far as I know, no other papers consider the depreciation rate as endogenous due to climate change. Greenwood et al. (1988) consider how endogenous capital depreciation, due to capacity utilization, affects the business cycle behavior.

In the models set up in this chapter, I will assume that climate affects productivity, the depreciation of capital and utility directly. I then analyze how the the different types of damages affect the behavior of the model. The models I use are modified neoclassical growth models. The factors of production are capital, labor, fossil-fuel based energy and alternative energy. The models build on the model of Golosov et al. (2011) which in turn builds on the models of Dasgupta and Heal (1974) and Stiglitz (1974).

The chapter can be seen as consisting of two parts. In the first part, I derive a formula for optimal taxation of fossil-fuel use in the context of an infinite time horizon model. In the second part of the chapter, I use a two-period model to study more in detail how the different types of damages affects the equilibrium fossil-fuel use.

The formula for the optimal tax that I derive, extends, under similar assumptions, the formula for the optimal tax from Golosov et al. (2011) to a situation where climate, in addition to affecting productivity, also affects utility directly and capital depreciation. Fossil-fuel use causes emissions of greenhouse gases which affects the future climate. The effects on the climate are not internalized in an unregulated market outcome and taxation of fossil-fuel use can be used to correct for this externality. The derived formula gives the optimal per unit tax as a constant times current total production in each period. The constant in the tax formula adds up the effects of each of the three forms of damages.

In order to derive this formula, I assume that utility from consumption of manufactured goods is logarithmic in the consumed amounts, that the climate state is a linear function of past emissions, that (old) capital depreciates fully and that the consumption (and savings) rate is constant. Furthermore, I assume that the effect of the climate state enters the utility function as an additive linear term and affects productivity and depreciation as a multiplicative term with constant elasticity. Apart from these assumptions, the model is very general in terms of the involved functions and parameters.

The derived formula can be used as a guide when trying to combine different kinds of damages into a joint damage function (as is done in, e.g., Nordhaus' DICE/RICE models).

After that, in the context of a two-period model, I investigate how the different types of damages affect fossil-fuel use. In the two-period model, I treat climate as exogenous and consider how equilibrium fossil-fuel use depends on productivity (in each of the periods), capital depreciation between the periods and on the climate state that enters the utility function (in both periods).

In this setting, I consider two different cases for the supply of fossil fuel. The first case is the oil case, where there are no extraction costs for fossil fuel and where the constraint on the total amount of available fossil fuel always binds. This can also be seen as the Hotelling case, where fossil-fuel use is completely driven by the mechanisms studied in the classic work by Hotelling (1931). In the second case, the coal case, scarcity does not matter at all, but extraction of fossil fuel uses capital and labor. These extraction costs are assumed to be independent of the remaining stock. These two cases can be seen as two extremes with reality lying somewhere in between.

The conclusion from this analysis is that both the assumptions about how the climate affects the economy and the assumption about fossil-fuel supply are very important for the effects of climate on equilibrium fossil-fuel use. One possible interpretation is that climate change increases capital depreciation, decreases productivity in the future (in the second period) or increases the marginal value of consumption of manufactured goods in the second period (due to a demand for adaptation measures). In this interpretation, climate effects will, within each of the cases, have opposite effects if it is assumed to affect capital depreciation compared to if it is assumed to affect productivity or utility. Comparing the two cases (oil and coal) the effects will, for each type of damage, be quite different.

I also make an attempt to assess whether the realization that there will be climate related effects will amplify or dampen climate change. To do this, I must determine what this means in each of the two cases. In the oil case, total fossil-fuel use is exogenously given and the division of the oil between the periods endogenously determined. I then interpret more first-period fossil-fuel use as an amplification of climate change. In the coal case, first-period fossil-fuel use is exogenously determined by the resources available in the first period, while second-period fossil-fuel use is endogenously determined. I then interpret more fossil-fuel use in the second period as amplification of climate change. Under this interpretation, climate effects on productivity or utility will dampen climate change in the oil case and amplify it in the coal case. Conversely, climate effects on the depreciation of capital will amplify climate change in the oil case, at least if the supply of alternative energy is exogenously given, but dampen it in the coal case.

The rest of the chapter is organized as follows. In the next section I set up the infinite time horizon model. I solve it both for the planner solution and for a decentralized equilibrium. I then derive the formula for the optimal taxation and make some simplifying assumptions that gives the tax formula a very simple form. After that, in section 3.3, I analyze how the different types of effects of climate change affect fossil-fuel use in the oil and coal case, respectively. The chapter is then concluded with a discussion of the results in section 3.4.

## 3.2 Model setup

The factors of production are capital $K$, labor $L$ and energy $E$. Capital is accumulated through investment while labor is exogenously given. Energy is a combination of fossil-fuel based and alternative energy. Generation of both types of energy can require the use of inputs. The use of fossil fuel is associated with emission of $CO_2$, which causes climate change. The climate state, $M$, determined by the history of fossil-fuel use, affects the productivity of final goods and the depreciation of capital and it also enters directly into the utility function.

Final goods production uses capital, labor and energy. It also depends on the climate state and on a technology factor $A_Y$. Let the amounts of inputs used in final goods production be $K_Y$, $L_Y$ and $E_Y$, respectively. I will further assume that the effect of the climate state on production enters as a multiplicative factor. Final goods production can then be written

$$Y = D_Y(M)F_Y(A_Y, K_Y, L_Y, E_Y), \tag{3.1}$$

where $D_Y$ gives the climate related effects on productivity, $A_Y$ is a technology factor and $F_Y$ is increasing in all its arguments and has constant returns to scale in $K$, $L$ and $E$.[1]

Energy comes from the burning of fossil fuel, which are extracted from the ground, and from alternative-energy generation. Let the remaining fossil-fuel resources at the beginning of period $t$ be $Q_t \geq 0$ and the amount extracted and burnt in period $t$ be $B_t \geq 0$. The remaining stock of fossil fuel and the fossil-fuel use in each period must fulfill

$$Q_{t+1} = Q_t - B_t \geq 0 \text{ and } B_t \geq 0. \tag{3.2}$$

Extraction may require the use of inputs. Let the amounts of inputs used in fossil-fuel extraction be $K_B$, $L_B$ and $E_B$. The extraction technology can also evolve over time. Let the state of the extraction technology be denoted $A_B$. Since the resources required for the extraction of fossil fuel may depend on the amount of remaining resources (if, e.g., the cheapest resources are used up first, then more resources are required to extract a given amount of fuels the smaller are the remaining reserves), the production of fossil fuel also depend on $Q$. In principle, climate change could also affect the extraction possibilities (for example by increasing the available resources due to ice melting in the arctic or by directly affecting the productivity in the extraction sector). I will not take the effect of climate on the extraction possibilities into account here.

Combined, this implies that the available amount of fossil-fuel based energy in a period can be written as

$$B = F_B(A_B, K_B, L_B, E_B, Q), \tag{3.3}$$

where $F_B$ is increasing in all its arguments.

Let the amount of alternative-energy use in a period be $S$. Let the state of the alternative-energy generation technology be $A_S$ and the inputs used in alternative-energy production be $K_S$, $L_S$ and $E_S$. These are then combined into the alternative energy according to the production function

$$S = F_S(A_S, K_S, L_S, E_S), \tag{3.4}$$

where $A_S$ is a technology factor and $F_S$ is increasing in all its arguments and has constant returns to scale in $K$, $L$ and $E$.

These two energy inputs are then combined into energy according to the production function

$$E = F_E(A_E, B, S), \tag{3.5}$$

[1] I will sometimes refer to $D_Y$ as damages. This can be a bit confusing since more damages is associated with a lower value of $D_Y$. The same thing goes for the climate effect on depreciation, which is introduced below and denoted $D_K$.

where $A_E$ is a technology factor and $F_E$ is increasing in all its arguments and has constant returns to scale in $B$ and $S$.

I will assume that total labor is exogenously given and equal to $L$.

The climate state depends on the past history of fossil-fuel use:

$$M_t = M(B_t, B_{t-1}, \ldots, B_0). \tag{3.6}$$

The dependency of climate on the history of emissions is, in reality, very complicated. The resulting increase in the concentration of $CO_2$ in the atmosphere, following emissions from the burning of fossil fuel, depends on the working of the carbon cycle. Increased concentrations of $CO_2$ in the atmosphere then changes the radiative balance leading to an increase in the temperature.

Below, I will make the assumption that the marginal effect of fossil-fuel use in period $t$ on the climate state in period $t' \geq t$ only depends on $t' - t$. This can be described by

$$\frac{dM_{t'}}{dB_t} = \begin{cases} \phi_{t'-t} & \text{if } t' \geq t \\ 0 & \text{if } t' < t \end{cases}. \tag{3.7}$$

While this ignores very much of the complexity of the climate system, it can be shown (see Golosov et al., 2011) that a model of this kind can replicate the behavior of the climate system in the DICE/RICE models well.

Capital is depreciated here for two reasons. Firstly, there is standard depreciation of capital given by a depreciation factor $\delta$. This leaves the resources that can be used for consumption or investment

$$Y_t + (1-\delta)K_t = C_t + I_t.$$

The investments are then depreciated further by climate change induced effects. This depreciation is given by the factor $D_{K,t} = D_K(M_t)$ and next period capital is

$$K_{t+1} = I_t D_{K,t} \Rightarrow I_t = \frac{K_{t+1}}{D_{K,t}}. \tag{3.8}$$

Consumption can then be written

$$C_t = Y_t + (1-\delta)K_t - \frac{K_{t+1}}{D_K(M_t)}. \tag{3.9}$$

In each period $t$, the inputs are divided between final goods production and production of fossil-fuel based or alternative energy. This gives the conditions

$$\begin{aligned} K_t &= K_{B,t} + K_{S,t} + K_{Y,t} \\ L_t &= L_{B,t} + L_{S,t} + L_{Y,t} \\ E_t &= E_{B,t} + E_{S,t} + E_{Y,t} \end{aligned} \tag{3.10}$$

that must hold for all $t$. I will assume that resources can be reallocated freely between periods so that the allocation of $K$, $L$ and $E$ is a static decision made in each period.

In each period $t$, utility derived in that period depends on the amount of consumption $C_t$ and on the climate state $M_t$. Total utility derived over the infinite time horizon is

$$\sum_{t=0}^{\infty} \beta^t U(C_t, M_t). \tag{3.11}$$

Thus, the climate related effects appear in three different places in the model: in productivity ($D_Y$), in capital depreciation ($D_K$) and in the utility function. While moderate climate change could potentially be beneficial in some places, the reasonable assumption is that, in a global model, climate change has negative consequences. This implies the derivative signs

$$D_Y'(M) \leq 0,\ D_K'(M) \leq 0 \text{ and } U_M(C, M) \leq 0.$$

The utility function also captures how climate change affects the marginal utility of consumption of the manufactured good. If the manufactured good can be used to adapt to adverse effects of climate change, the reasonable sign of the cross derivative is

$$U_{CM} \geq 0$$

This completes the description of the model. I will now solve the model both for the planner solution and for a decentralized equilibrium with taxation of fossil fuel. Since there are climate externalities, the planner solution and the decentralized equilibrium will, in general, not coincide and there will be scope for improving on the decentralized equilibrium using taxation.

### 3.2.1 Planner solution

The planner is assumed to want to maximize welfare and be restricted only by the physical constraints on the economy. This means that the planner wants to maximize (3.11) given all the conditions stated above. The set of variables that the planner chooses is

$$\left\{Y_t, C_t, K_{t+1}, E_t, B_t, S_t, \{K_{X,t}, L_{X,t}, E_{X,t}\}_{X\in\{B,S,Y\}}, Q_{t+1}, M_t\right\}_{t=0}^{\infty}. \tag{3.12}$$

In principle, all variables should be subject to non-negativity constraints. Except for the case of $Q_t$, I will assume that the non-negativity constraints never bind. I will therefore exclude these from the formulation of the problem. This presumption would, for example, be satisfied if all the production functions fulfilled Inada conditions with respect to all inputs, that is, if the marginal product of an input, in all sectors, goes to infinity when use of the input goes to zero. In the example in section 3.2.5 I will assume that energy production does not use capital. That means that the Inada conditions are not fulfilled there, but it is straightforward to adapt the equilibrium conditions to that case.

I will not set up the planner problem explicitly. Instead I will write down the Lagrangian of the problem, which is given by

$$
\begin{aligned}
\mathcal{L} = & \sum_{t=0}^{\infty} \beta^t U(C_t, M_t) + \sum_{t=0}^{\infty} \lambda_{C,t} \left[ Y_t + (1-\delta) K_t - \frac{K_{t+1}}{D_K(M_t)} - C_t \right] \\
& + \sum_{t=0}^{\infty} \lambda_{Y,t} \left[ D_Y(M_t) F_Y(A_{Y,t}, K_{Y,t}, L_{Y,t}, E_{Y,t}) - Y_t \right] \\
& + \sum_{t=0}^{\infty} \lambda_{E,t} \left[ F_E(A_{E,t}, B_t, S_t) - E_t \right] \\
& + \sum_{t=0}^{\infty} \lambda_{B,t} \left[ F_B(A_{B,t}, K_{B,t}, L_{B,t}, E_{B,t}, Q_t) - B_t \right] \\
& + \sum_{t=0}^{\infty} \lambda_{S,t} \left[ F_S(A_{S,t}, K_{S,t}, L_{S,t}, E_{S,t}) - S_t \right] \\
& + \sum_{t=0}^{\infty} \sum_{Z \in \{K,L,E\}} \mu_{Z,t} \left[ Z_t - \sum_{X \in \{B,S,Y\}} Z_{X,t} \right] \\
& \sum_{t=0}^{\infty} \lambda_{M,t} \left[ M_t(B_t, B_{t-1}, \ldots, B_0) - M_t \right] \\
& \sum_{t=0}^{\infty} \lambda_{Q,t} \left[ Q_{t+1} + B_t - Q_t \right] + \sum_{t=0}^{\infty} \mu_{Q,t} Q_t,
\end{aligned}
$$

where all the $\mu$s and $\lambda$s are multipliers.

Taking first-order conditions with respect to each of the variables gives

$$Y_t : \lambda_{C,t} = \lambda_{Y,t} \tag{3.13}$$
$$C_t : \lambda_{C,t} = \beta^t U_{C,t} \tag{3.14}$$
$$K_{t+1} : \lambda_{C,t} = D_K(M_t)\left[\lambda_{C,t+1}(1-\delta) + \mu_{K,t+1}\right] \tag{3.15}$$
$$E_t : \lambda_{E,t} = \mu_{E,t} \tag{3.16}$$
$$B_t : \lambda_{B,t} = \lambda_{E,t} F_{B,E,t} + \lambda_{Q,t} + \sum_{t'=t}^{\infty} \lambda_{M,t} \frac{dM_{t'}}{dB_t} \tag{3.17}$$
$$S_t : \lambda_{S,t} = \lambda_{E,t} F_{E,S,t}$$
$$Q_{t+1} : \lambda_{Q,t+1} = \lambda_{Q,t} + \lambda_{B,t+1} F_{B,Q,t+1} + \mu_{Q,t+1} \tag{3.18}$$
$$M_t : \lambda_{M,t} = \beta^t U_{M,t} + \lambda_{C,t} \frac{K_{t+1}}{D_K(M_t)} \frac{D'(M_t)}{D_K(M_t)} + \lambda_{Y,t} Y_t \frac{D'_Y(M_t)}{D_Y(M_t)} \tag{3.19}$$

For $Z \in \{K, L, E\}$, the first order conditions with respect to $Z_{Y,t}$, $Z_{B,t}$ and $Z_{S,t}$ give that

$$\mu_{Z,t} = \lambda_{Y,t} D_Y(M_t) F_{Y,Z,t} = \lambda_{B,t} F_{B,Z,t} = \lambda_{S,t} F_{B,S,t}. \tag{3.20}$$

These expressions state that each of the inputs capital, labor and energy should be allocated across sectors so that their marginal values in each sector are equalized.

Combining (3.14) and (3.15) gives the capital accumulation condition

$$\beta^t U_{C,t} = \left[\mu_{K,t+1} + \beta^{t+1} U_{C,t+1}(1-\delta)\right] D_K(M_t).$$

This equation is similar to the standard Euler equation derived in a one-sector model. There are two differences. The first difference is the factor $D_K(M_t)$, which captures that a share of any investment is lost due to climate related damages. This lowers the returns to investment. The second difference is the term $\mu_{K,t+1}$ appearing where there is usually a marginal product of capital in period $t+1$. From equation (3.20) in period $t+1$, and with $Z = K$, it follows that

$$\mu_{K,t+1} = \lambda_{Y,t+1} D_Y(M_{t+1}) F_{Y,K,t+1} = \lambda_{B,t+1} F_{B,K,t+1} = \lambda_{S,t+1} F_{S,K,t+1},$$

where each of these expressions is the shadow value of the output in the sector times the marginal product of capital in that sector. That is, $\mu_{K,t+1}$ gives the marginal value of the marginal product of capital in each of the sectors.

From (3.13) and (3.14), the shadow value of final goods production in period $t+1$ is $\lambda_{Y,t+1} = \beta^{t+1} U_{C,t+1}$. Substituting this expression into the capital accumulation condition gives

$$\beta^t U_{C,t} = \beta^{t+1} U_{C,t+1}\left[D_Y(M_{t+1}) F_{Y,K,t+1} + 1 - \delta\right] D_K(M_t). \tag{3.21}$$

Turning now, instead, to the fossil-fuel use decision, the shadow value of fossil-fuel use in period $t$ is $\lambda_{B,t}$ and it is given in (3.17). It can be divided into three parts. The first term consists of the shadow value of energy times the marginal product of fossil fuel in producing energy: the marginal value of using fossil fuel in production. The second term is the shadow value of the remaining stock. From (3.18) it can be seen that this shadow value captures the scarcity value of the resource (coming from the finiteness of the resource) and the dependency of the extraction costs on the remaining stock. Each of these factors gives a value to keeping fuels in the ground. The third term in (3.17) captures the marginal cost of the climate damages caused from period $t$ and onwards caused by emissions in period $t$. It is the sum, from $t$ to infinity, of the shadow value of the climate state in that period, times the effect on the climate state of the emissions in period $t$.

Substituting for $\lambda_{Y,t}$ and $\lambda_{C,t}$ from (3.13) and (3.14), respectively, and for $\frac{K_{t+1}}{D_K(M_t)}$ from (3.8) into (3.19), the shadow value of the climate state becomes

$$\lambda_{M,t'} = \beta^{t'} \left[ U_{M,t'} + U_{C,t'} \left( I_{t'} \frac{D'_K(M_{t'})}{D_K(M_{t'})} + Y_{t'} \frac{D'_Y(M_{t'})}{D_Y(M_{t'})} \right) \right].$$

The right-hand side captures the three effects that the climate is assumed to have. The first term captures the direct effect on welfare, the second term captures the effect on the capital stock and the third term captures the effect on productivity. The effects are computed as marginal utility effects in period $t'$ discounted back to period 0.

Using (3.20), with $Z = E$, and (3.16), the shadow value of energy is

$$\lambda_{E,t} = \lambda_{Y,t} D_Y(M_t) F_{Y,E,t} = \{(3.14) \text{ and } (3.15)\} = \beta^t U_{C,t} D_Y(M_t) F_{Y,E,t}.$$

Substituting for $\lambda_{M,t'}$ and $\lambda_{E,t}$ in (3.17) gives the fossil-fuel use condition

$$\lambda_{B,t} = \sum_{t'=t}^{\infty} \beta^{t'} \left[ U_{M,t'} + U_{C,t'} \left( I_{t'} \frac{D'_K(M_{t'})}{D_K(M_{t'})} + Y_{t'} \frac{D'_Y(M_{t'})}{D_Y(M_{t'})} \right) \right] dB_t + \beta^t U_{C,t} D_Y(M_t) F_{Y,E,t} F_{E,B,t} + \lambda_{Q,t}. \tag{3.22}$$

This concludes the characterization of the solution to the planner problem. The two most important conditions are the capital accumulation condition (3.21) and the fossil-fuel use condition (3.22).

### 3.2.2 Decentralized equilibrium with taxation

In the decentralized equilibrium, the decisions are made by different actors. Households are assumed to derive income from renting out capital to firms, from renting out labor to firms, from profits in firms that they own shares in and from government lump-sum transfers. There are four different types of firms in the model: final goods producing firms, fossil-fuel extracting firms, alternative-energy producing firms and firms that combine the output from the two energy sources into the composite energy good. Let the profits of these firms be $\pi_Y$, $\pi_B$, $\pi_S$ and $\pi_E$ respectively. I will assume that all firms are owned in equal shares by all households.

The government taxes sales of fossil fuel with a per unit tax $\hat{\tau}$ and pays lump-sum transfers $g$ to the households.

I will assume that all agents act as price takers. I will normalize the price of final goods in each period to 1. In addition to this, there will be 5 prices. Let the energy price be $p_E$, the price of labor be $w$, the rental rate of capital be $r$, the price of fossil-fuel based energy be $p_B$ and the price of alternative energy be $p_S$.

A competitive equilibrium can now be defined as a sequence of prices $\{p_{E,t}, w_t, r_t, p_{B,t}, p_{S,t}\}_{t=0}^{\infty}$, quantities (3.12) and taxes and government transfers $\{\hat{\tau}_t, g_t\}_{t=0}^{\infty}$ such that

- All firms maximize profits
- Households maximize utility
- The government budget is balanced in each period
- The climate state is given by (3.6)

I will first solve the firms' profit maximization problems and then the households' utility maximization problem. After that, I will combine these conditions, and the government's balanced budget constraint, to characterize the equilibrium allocation. I will not explicitly need to use the equation for the climate state. In this section, I will treat the taxes as given. In section 3.2.3 I will then derive the taxes that implement the planner solution.

### Firms

Final good producing firms, alternative-energy producing firms and composite energy producing firms all rent or buy their inputs in each period and therefore face a static problem in each period. Under assumptions of perfect competition and constant returns to scale (in capital, labor, and energy for final goods and alternative-energy producing firms, and to fossil-fuel based and alternative energy for composite energy producing firms), each firm will use each input up to the point where the value of the marginal product of the input equals its price. This gives the equilibrium conditions[2]

$$r_t = D_Y(M_t)F_{Y,K,t} = p_{S,t}F_{S,K,t} \tag{3.23}$$
$$w_t = D_Y(M_t)F_{Y,L,t} = p_{S,t}F_{S,L,t} \tag{3.24}$$
$$p_{E,t} = D_Y(M_t)F_{Y,E,t} = p_{S,t}F_{S,E,t} \tag{3.25}$$
$$p_{B,t} = p_{E,t}F_{E,B,t} \tag{3.26}$$
$$p_{S,t} = p_{E,t}F_{E,S,t}.$$

Under the constant returns to scale assumptions, final goods, alternative energy and composite energy producing firms will all make zero profits:

$$\forall t: \pi_{Y,t} = \pi_{S,t} = \pi_{E,t} = 0.$$

The fossil-fuel extracting firms own fossil-fuel resources and therefore must take the effect of extraction on the remaining stock into account. This makes their profit maximization problem dynamic. They choose an extraction path to maximize the discounted sum of profits.

The profit made, in each period, is the after tax revenues from the sales of fossil fuel minus the costs for the inputs used in extraction:

$$\pi_{B,t} = (p_{B,t} - \hat{\tau}_t)\, B_t - r_t K_{B,t} - w_t L_{B,t} - p_{E,t}E_{B,t}. \tag{3.27}$$

The discount rate between two periods is given by the net return to investment in capital, that is, the return that profits invested in capital would yield. Define the discount factors

$$R_{t+1} = D_K(M_t)\left[r_{t+1} + 1 - \delta\right] \text{ and } R_{t_1,t_2} = \begin{cases} 1 & \text{if } t_1 = t_2 \\ \prod_{t=t_1+1}^{t_2} R_t & \text{if } t_2 > t_2 \end{cases},$$

where $R_{t+1}$ gives the return in period $t+1$ from investments made in period $t$ and $R_{t_1,t_2}$ gives the discount factor in period $t_1$ for income derived in period $t_2 \geq t_1$.

The fossil-fuel extracting firms are thus assumed to maximize the discounted profit stream. For simplicity, I will assume that the fossil-fuel resources consists of large number of identical sources owned by different firms. Each firm solves the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \frac{1}{R_{0,t}} \left[(p_{B,t} - \tau_t)\, B_t - r_t K_{B,t} - w_t L_{B,t} - p_{E,t}E_{B,t}\right] \\ &\text{s.t.} \quad \forall t: B_t = F_B(A_{B,t}, K_{B,t}, L_{B,t}, E_{B,t}, Q_t) \\ &\qquad Q_{t+1} = Q_t - B_t \geq 0;\ B_t \geq 0, \end{aligned}$$

[2]As in the case of the planner solution, I will assume interior solutions.

where the maximization is over $\{B_t, Q_{t+1}, K_{B,t}, L_{B,t}, E_{B,t}\}_{t=0}^{\infty}$. The Lagrangian of this problem is

$$\begin{aligned}\mathcal{L} = & \sum_{t=0}^{\infty} \frac{1}{R_{0,t}} \left[ (p_{B,t} - \tau_t) B_t - r_t K_{B,t} - w_t L_{B,t} - p_{E,t} E_{B,t} \right] \\ & + \sum_{t=0}^{\infty} \lambda_{B,t} \left[ F_B(A_{B,t}, K_{B,t}, L_{B,t}, E_{B,t}, Q_t) - B_t \right] \\ & + \sum_{t=0}^{\infty} \lambda_{Q,t} \left[ Q_{t+1} + B_t - Q_t \right] + \sum_{t=1}^{\infty} \mu_{Q,t} Q_t.\end{aligned}$$

The first-order conditions read

$$B_t : \lambda_{B,t} = \frac{1}{R_{0,t}} (p_{B,t} - \tau_t) + \lambda_{Q,t} \tag{3.28}$$

$$\begin{aligned} Q_{t+1} &: \lambda_{Q,t+1} = \lambda_{Q,t} + \lambda_{B,t+1} F_{B,Q,t+1} + \mu_{Q,t+1} \\ K_{B,t} &: \frac{1}{R_{0,t}} r_t = \lambda_{B,t} F_{B,K,t} \\ L_{B,t} &: \frac{1}{R_{0,t}} w_t = \lambda_{B,t} F_{B,L,t} \\ E_{B,t} &: \frac{1}{R_{0,t}} p_{E,t} = \lambda_{B,t} F_{B,E,t}. \end{aligned}$$

I have now derived the profit maximization condition for all types of firms.

### Households

In each period, households derive income from capital, labor, profits from firms they own shares in and government transfers. The only firms that will make profits are the fossil-fuel extracting firms. Furthermore, I will not model trade in shares in firms. In a representative household model, there can, in equilibrium, be no trade in any shares. Including trade in shares would simply determine the equilibrium prices of those shares.

The households' budget constraint is

$$C_t + \frac{K_{t+1}}{D_K(M_t)} = (r_t + 1 - \delta) K_t + w_t L_t + \pi_{B,t} + g_t. \tag{3.29}$$

Households solve the utility maximization problem

$$\begin{aligned} \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t U(C_t, M_t) \\ \text{s.t. } & (3.29)\ \forall t. \end{aligned}$$

The Lagrangian of this problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(C_t, M_t) + \sum_{t=0}^{\infty} \lambda_{C,t} \left[ (r_t + 1 - \delta) K_t + w_t L_t + \pi_t + T_t - C_t - \frac{K_{t+1}}{D_K(M_t)} \right].$$

The first-order conditions of this problem are

$$C_t : \lambda_{C,t} = \beta^t U_{C,t}$$
$$K_{t+1} : \lambda_{C,t} = \lambda_{C,t+1} \left[ r_{t+1} + 1 - \delta \right] D_K(M_t).$$

Combining these two conditions delivers the households' Euler equation:

$$\frac{U_{C,t}}{\beta U_{C,t+1}} = D_K(M_t) \left[ r_{t+1} + 1 - \delta \right]. \tag{3.30}$$

This concludes the solution of the households' utility maximization problem.

**Government**

In each period, the government receives tax revenues $\hat{\tau}_t B_t$ and pays lump-sum transfers $g_t$. The requirement that the government's budget must be balanced in each period gives that

$$g_t = \hat{\tau}_t B_t. \tag{3.31}$$

**Allocation**

I have now derived all the equilibrium conditions and will combine them to characterize the equilibrium allocation. In particular, I will derive the capital accumulation and fossil-fuel use conditions. I will also verify that all income in the economy goes to the households so that the households' budget constraint, in equilibrium, coincides with the resource constraint of the economy.

Substituting for the price of capital (3.23) in the Euler equation (3.30) gives

$$U_{C,t} = \beta U_{C,t+1} D_K(M_t) \left[ D_Y(M_{t+1}) F_{Y,K,t+1} + 1 - \delta \right]. \tag{3.32}$$

Comparing this to condition (3.21), they are the same.

Substituting (3.25) into (3.26), the fossil-fuel price in period $t$ becomes

$$p_{B,t} = p_{E,t} F_{E,B,t} = D_Y(M_t) F_{Y,E,t} F_{E,B,t}. \tag{3.33}$$

Using the households' Euler equation (3.30), the discount factor is

$$R_{0,t} = \prod_{t'=1}^{t} D_k(M_{t'-1}) \left[r_{t'} + 1 - \delta\right] = \prod_{t'=1}^{t} \frac{U_{C,t'-1}}{\beta U_{C,t'}} = \frac{U_{C,0}}{\beta^t U_{C,t}}. \qquad (3.34)$$

Substituting the fossil-fuel price (3.33) and the discount factor (3.34) into the fossil-fuel owners' first-order condition with respect to $B_t$ (3.28) and rewriting gives

$$U_{C,0}\lambda_{B,t} = \beta^t U_{C,t} D_Y(M_t) F_{Y,E,t} F_{E,B,t} - \beta^t U_{C,t}\tau_t + U_{C,0}\lambda_{Q,t}. \qquad (3.35)$$

Comparing this equation to the fossil-fuel use condition from the planner solution (3.22), they are in some ways similar, but there are some important differences. The effect of fossil-fuel use on the climate is not at all internalized here. In the place of the value of fossil-fuel use in production there is now the net of tax profit from fossil-fuel use. A less important difference is that the multipliers are here multiplied by $U_{C,0}$. Apart from that, the shadow value of the stock of fossil fuel enters the condition in much the same way as in (3.22). The comparison between (3.22) and (3.35) is what will give the expression for the optimal tax in section 3.2.3.

I will now verify that all income goes to the households. Substituting for the profit from the fossil-fuel extracting firms (3.27) and the lump-sum transfer (3.31) in the households' budget constraint (3.29) gives

$$\begin{aligned} C_t + \frac{K_{t+1}}{D_K(M_t)} &= (r_t + 1 - \delta)K_t + w_t L_t \\ &\quad + \left(p_{B,t} - \hat{\tau}_t\right) B_t - r_t K_{B,t} - w_t L_{B,t} - p_{E,t} E_{B,t} + \hat{\tau}_t B_t \\ &= \left(K_t - K_{B,t}\right) r_t + \left(L_t - L_{B,t}\right) w_t + p_{B,t} B_t \\ &\quad - p_{E,t} E_{B,t} + (1 - \delta) K_t. \end{aligned}$$

Using that alternative-energy generating firms make zero profits, we obtain

$$p_{S,t} S_t = r_t K_{S,t} + w_t L_{S,t} + p_{E,t} E_{S,t}.$$

Similarly, using the fact that composite energy producing firms make zero profits, one arrives at

$$\begin{aligned} p_{E,t} E_t &= p_{S,t} S_t + p_{B,t} B_t \\ &= r_t K_{S,t} + w_t L_{S,t} + p_{E,t} E_{S,t} + p_{B,t} B_t. \end{aligned}$$

This delivers

$$p_{B,t} B_t = p_{E,t} E_t - r_t K_{S,t} - w_t L_{S,t} - p_{E,t} E_{S,t}.$$

Substituting this into the households' budget constraint gives

$$
\begin{aligned}
C_t + \frac{K_{t+1}}{D_K(M_t)} &= (K_t - K_{B,t})\, r_t + (L_t - L_{B,t})\, w_t \\
&\quad + p_{E,t}E_t - r_t K_{S,t} - w_t L_{S,t} - p_{E,t}E_{S,t} \\
&\quad - p_{E,t}E_{B,t} + (1-\delta)K_t \\
&= (K_t - K_{B,t} - K_{S,t})\, r_t + (L_t - L_{B,t} - L_{S,t})\, w_t \\
&\quad + (E_t - E_{B,t} - E_{S,t})\, p_{E,t} + (1-\delta)K_t \\
&= K_{Y,t}r_t + L_{Y,t}w_t + E_{Y,t}p_{E,t} + (1-\delta)K_t \\
&= \{(3.23),\ (3.24)\ \text{and}\ (3.25)\} \\
&= D_Y(M_t)\left(F_{Y,K,t}K_{Y,t} + F_{Y,L,t}L_{Y,t} + F_{Y,E,t}K_{E,t}\right) \\
&\quad + (1-\delta)K_t \\
&= D_Y(M_t)F_{Y,t} + (1-\delta)K_t \\
&= Y_t + (1-\delta)K_t.
\end{aligned}
$$

The resulting equation is the resource constraint from the planner solution (3.9).

This concludes the characterization of the decentralized equilibrium. The most important conditions are the capital accumulation condition (3.32) and fossil-fuel use condition (3.35).

### 3.2.3 Optimal taxation

I will now derive a formula for taxes that, in the decentralized equilibrium with taxation, allows implementation of the planner solution.

As noted above, comparing the Euler equations from the planner solution (3.21) and the decentralized equilibrium (3.32) they are the same. Comparing the fossil-fuel use conditions (3.22) and (3.35), the taxes $\{\hat{\tau}_t\}_{t=0}^{\infty}$ should be chosen to make the two conditions equivalent. This can be achieved by in each period $t$ choosing the per-unit tax

$$
\hat{\tau}_t = -\sum_{t'=t}^{\infty}\left[\frac{\beta^{t'}U_{C,t'}}{\beta^t U_{C,t}}\left(I_{t'}\frac{D_K'(M_{t'})}{D_K(M_{t'})} + Y_{t'}\frac{D_Y'(M_{t'})}{D_Y(M_{t'})}\right) + \frac{\beta^{t'}U_{M,t'}}{\beta^t U_{C,t}}\right]\frac{dM_{t'}}{dB_t}, \tag{3.36}
$$

all evaluated at the allocation from the planner solution.

It can also be shown that all the conditions for intratemporal allocation of resources between final goods production, fossil-fuel extraction and alternative-energy generation are the same in both cases.

So, if the tax is set according to this formula, the conditions for the planner solution and the decentralized equilibrium are equivalent. Therefore, the taxes given by (3.36) implement the planner solution. The expression within the square bracket in (3.36) is the sum of the three types of marginal damages in period $t'$, caused by a change in the climate state, $M_{t'}$, expressed in terms of period $t$ consumption. This is then multiplied by the marginal effect of emissions in period $t$ on the climate state in period $t'$ and summed over $t' \geq t$. Thus, the right-hand side (3.36) gives total discounted marginal damages caused by emissions in period $t$.

### 3.2.4 Simplifying assumptions

I will now make some simplifying assumptions that allow me to derive a very simple formula for the tax.

The utility function will be assumed to have the following form:

$$U(C, M) = \ln(C) - \kappa_U M. \tag{3.37}$$

I assume that the damages to the capital stock and to productivity have the following forms:

$$D_Y(M) = e^{-\kappa_Y M} \text{ and } D_K(M) = e^{-\kappa_K M}. \tag{3.38}$$

I also assume that the carbon cycle is linear so that the effect of emissions in period $t$, on the future state of the climate, is given by (3.7). As shown by Golosov et al. (2011), this specification, with climate only affecting productivity, can replicate the climate systems of the DICE/RICE models well.

Under these assumptions, the optimal per unit tax becomes

$$\hat{\tau}_t = C_t \sum_{t'=t}^{\infty} \beta^{t'-t} \left[ \frac{I_{t'}}{C_{t'}} \kappa_K + \frac{Y_{t'}}{C_{t'}} \kappa_Y + \kappa_U \right] \phi_{t'-t}.$$

The tax can be rewritten using the consumption rate

$$c_t = \frac{C_t}{Y_t}.$$

Using the consumption rate, the ratio of investment to consumption is

$$C_t + I_t = Y_t + (1-\delta)K_t \Rightarrow$$
$$\tfrac{I_t}{C_t} = \tfrac{Y_t}{C_t} - 1 + (1-\delta)\tfrac{K_t}{C_t} = \tfrac{1-c_t}{c_t} + (1-\delta)\tfrac{K_t}{c_t Y_t}.$$

Assuming further that there is full depreciation ($\delta = 1$)

$$\frac{I_t}{C_t} = \frac{1-c_t}{c_t}.$$

The per unit tax then becomes

$$\hat{\tau}_t = c_t Y_t \sum_{t'=t}^{\infty} \beta^{t'-t} \left[ \frac{1-c_{t'}}{c_{t'}} \kappa_K + \frac{1}{c_{t'}} \kappa_Y + \kappa_U \right] \phi_{t'-t}. \tag{3.39}$$

Assuming, finally, that the consumption rate is constant (in the planner solution), $c_t = c$ for all $t$, then the following proposition can be stated.

**Proposition 3.1.** *Assume that the utility function and the damage functions are given by* (3.37) *and* (3.38)*. Assume further that* $\delta = 1$*, that the carbon cycle fulfills* (3.7) *and that the consumption rate* $c$ *is constant. Then the per unit tax in period* $t$ *should be*

$$\hat{\tau}_t = Y_t \left[(1-c)\kappa_K + \kappa_Y + c\kappa_U\right] \sum_{t'=t}^{\infty} \beta^{t'-t} \phi_{t'-t} \tag{3.40}$$

*That is, the optimal tax is a constant times production in that period.*

*Proof.* Follows from (3.39) with $c_t = c$ for all $t$. □

So, in each time period, the per unit tax should be a (time-invariant) constant times production in that period. Comparing this formula to the corresponding formula in Golosov et al. (2011) they are the same if $\kappa_K = \kappa_U = 0$. If $\kappa_K \neq 0$ or $\kappa_U \neq 0$ the tax, unlike in Golosov et al., depends on the value of the consumption rate. Note also that I here need to assume full depreciation to get the tax as a constant times production. The reason for this is that both the consumption and the savings rate (as shares of production) need to be constant, which is only possible when there is full depreciation.

The consumption rate enters the calculations just made in two different ways. Firstly, it matters for the marginal effects in period $t'$ and secondly, it matters for the marginal utility of consumption in period $t$. Starting with period $t'$, the effects in terms of period $t'$ marginal utility are given within the square brackets in equation (3.39). The amount of resources destroyed by the damages to capital depreciation is proportional to investment. The damages are expressed in utility terms through multiplication by the marginal utility of consumption leaving the ratio $\frac{1-c}{c}$. Both the numerator and denominator of this ratio implies a negative dependency on $c$. The amount of resources destroyed by damages to production is proportional to production and these are expressed in utility terms through multiplication by the marginal utility of consumption resulting in the ratio $\frac{1}{c}$. The damages to utility are already expressed in utility terms. When translating the effects in period $t'$ into period $t$ consumption, this is achieved through division by period $t$ marginal utility of consumption. This step results in the initial factor $c_t Y_t$ in the right-hand side of (3.39). Thus, in relation to $Y_t$ this last step gives a positive dependency of the tax on the consumption rate. The net dependencies of the different terms in (3.40) on the consumption rate are the net effects. This net effect is positive for damages to utility, zero for damages to productivity (since the dependency on $c$ from the two steps cancel) and negative for damages to depreciation.

### 3.2.5 A closed-form example

I will now provide an example where the conditions of proposition 3.1 are fulfilled. I will assume that the utility function and the damage functions are given by (3.37) and (3.38) and that there is full depreciation $\delta = 1$. Furthermore, I will assume that capital is not used in either fossil-fuel extraction or in alternative-energy generation and that capital enters the final goods production function as in a Cobb-Douglas production function. The last two assumptions imply that

$$K_{Y,t} = K_t,\ K_{B,t} = K_{S,t} = 0 \text{ for all } t$$

and

$$Y_t = D_Y(M_t)F_{Y,t} = D_Y(M_t)\bar{F}_{Y,t}K_t^{\bar{\alpha}},$$

where $\bar{\alpha}$ is a given constant and where $\bar{F}_{Y,t}$ may depend on all of the variables that $F_{Y,t}$ depends on except $K_t$.

With $F_{Y,t}$ given like this, the marginal product of capital is

$$D_Y(M_t)F_{Y,K,t} = \bar{\alpha}D_Y(M_t)\bar{F}_{Y,t}K_t^{\bar{\alpha}-1} = \bar{\alpha}\frac{Y_t}{K_t}.$$

Substituting this into the Euler equation of the planner solution (3.21) and using the fact that utility is logarithmic, we obtain

$$\frac{1}{C_t} = \beta\frac{1}{C_{t+1}}\bar{\alpha}\frac{Y_{t+1}}{K_{t+1}}D_K(M_t).$$

With full depreciation, period $t$ investment is

$$I_t = \frac{K_{t+1}}{D_K(M_t)}.$$

The Euler equation can now be written

$$\frac{I_t}{C_t} = \bar{\alpha}\beta\frac{Y_{t+1}}{C_{t+1}}.$$

This is solved, in each period, by setting

$$C_t = (1-\bar{\alpha}\beta)Y_t \text{ and } I_t = \bar{\alpha}\beta Y_t.$$

That is, the consumption rate is constant and equal to $1-\bar{\alpha}\beta$. Using proposition 3.1, the tax should, in each period, be given by

$$\tau_t = Y_t\left[\bar{\alpha}\beta\kappa_K + \kappa_Y + (1-\bar{\alpha}\beta)\kappa_U\right]\sum_{t'=t}^{\infty}\beta^{t'-t}\phi_{t'-t}.$$

So, the optimal tax can be calculated without actually solving for the optimal path of fossil-fuel use and within-period allocation of inputs.

## 3.3 Two-period model

In this section, I will consider a two-period version of the model, to see how changes in the variables affected by climate change ($D_K$, $D_Y$ and $M$) changes fossil-fuel use. I will treat the "climate" as exogenously given and then derive the effects of varying the variables.

I will consider two different cases for the extraction of fossil fuel. The first case will be the "oil" case where fossil fuel is costlessly extracted from a given total supply. The second case will be the "coal" case where the finiteness of the resource does not matter and where extraction uses resources.

The first period is the same as in the infinite horizon model. First period production is $Y_1$. This is divided between consumption and investment. The first period capital $K_1$ is given. This capital depreciates at the rate $\delta$. Investments made in period 1, and the remaining capital, is subjected to climate related damages after which the share $D_K$ remains.

$$K_2 = (Y_1 - C_1 + (1-\delta)K_1)\, D_K \Rightarrow C_1 = Y_1 + (1-\delta)K_1 - \frac{K_2}{D_K}.$$

Second-period production is consumed:

$$C_2 = Y_2.$$

There is a climate related damage that affects productivity in each period, $D_{Y,t}$, and a climate state, $M_t$, that affects utility directly.

Regarding the one-period utility function, I will now assume that, for given $M$, the utility function behaves as a standard CES-utility function in the sense that

$$\frac{U_{CC}}{U_C} = -\theta \frac{1}{C} \tag{3.41}$$

and that, as before,

$$U_{CM} \geq 0$$

so that more climate change increases the marginal utility from consuming manufactured goods. The particular utility function defined in (3.37) fulfills the last condition with equality.

I will now also be specific about the production function. Capital and labor is aggregated according to a Cobb-Douglas production function into a composite $G$

$$G = K^{\alpha} L^{1-\alpha}. \tag{3.42}$$

Fossil fuel, $B$, and alternative energy, $S$, are combined, according to a CES production function with elasticity $\sigma_E$, into the composite energy good $E$:

$$E = F_E(B, S) = \left[\gamma_B B^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S S^{\frac{\sigma_E - 1}{\sigma_E}}\right]^{\frac{\sigma_E}{\sigma_E - 1}}. \tag{3.43}$$

The composites $G$ and $E$ are then aggregated into final goods according to a CES-function with elasticity $\sigma_Y$:

$$Y = D_Y F_Y(G, E) = D_Y \left[\gamma_G G^{\frac{\sigma_Y - 1}{\sigma_Y}} + \gamma_E E^{\frac{\sigma_Y - 1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y - 1}}. \tag{3.44}$$

For notational simplicity, I do not include any productivities $A_Y$ or $A_E$ here. Including those as multiplicative terms would not change the essence of the analysis that is to follow.

In the oil and coal cases below, I will make different assumptions about how $B$ and $S$ are produced. I will treat the climate as exogenous. I will then vary the climate related variables to see how different possible effects of climate change will affect equilibrium fossil-fuel use.

The results will depend on the values of the elasticities in the utility and production functions. I therefore need to make some assumptions about those parameters.

Above I assumed that utility was logarithmic, that is, $\theta = 1$. This is likely to be a reasonable assumption; Layard et al. (2008) find a value of $\theta$ between 1.2 and 1.3.

There is low substitutability between energy and other inputs in the short run. Hassler et al. (2011) estimate the elasticity to be 0.005. This implies that $\sigma_Y$ should be small. The longer a time period is chosen to be, the larger the substitutability can be expected to be since more adjustments, e.g., more investments in energy efficiency, are possible. Assuming limited substitutability still seems reasonable.

The substitutability between different energy sources is high. If energy is converted into electricity, the original source of the energy is irrelevant and there is perfect substitutability. For other energy uses, e.g., cars, investment in new machines may be required, but once these are made, substitutability is high. In other uses, e.g., for flying planes, fossil fuels seem much more difficult to substitute away. Combined, this seems to imply that, at the aggregate level, different energy sources are good, but not perfect, substitutes.

Based on all this, the following assumptions about parameters seem reasonable

$$0 < \frac{1}{\sigma_E} \leq 1 \leq \theta \leq \frac{1}{\sigma_Y} < \infty \tag{3.45}$$

### 3.3.1 The oil case

In this case, fossil fuel is costlessly extracted from a total supply $Q$. I will below first treat alternative energy as exogenously given. After that, I will assume that alternative-energy generation requires labor and capital. The allocation of labor and capital between final goods production and alternative-energy generation will be a static decision that can be made within the period.

This means that production in each period depends on capital and amounts of fossil fuel used. That is, production can then be written as

$$Y_t = D_{Y,t}\tilde{F}_Y(B_t, K_t).$$

**Planner solution**

Since I assume that the climate related parameters are exogenously given, the planner solution and decentralized equilibrium will coincide and I will solve for the planner solution. All fossil fuel will be exhausted. Given parameter assumptions (3.45), fossil fuel will be used in both periods and in the second period whatever amount was not used in the first period will now be used:

$$B_2 = Q - B_1.$$

This reduces the problem to choosing first-period fossil-fuel use and second-period capital. The planner problem is

$$\max_{B_1,K_2} \begin{bmatrix} U\left(D_{Y,1}\tilde{F}_Y(K_1,B_1) + (1-\delta)K_1 - \frac{K_2}{D_K}, M_1\right) \\ +\beta U\left(D_{Y,2}\tilde{F}_Y(K_2, Q - B_1), M_2\right) \end{bmatrix}.$$

The first-order conditions read

$$\begin{aligned} B_1 &: U_{C,1}D_{Y,1}\tilde{F}_{Y,B,1} = \beta U_{C,2}D_{Y,2}\tilde{F}_{Y,B,2} \\ K_2 &: U_{C,1}\frac{1}{D_K} = \beta U_{C,2}D_{Y,2}\tilde{F}_{Y,K,2}. \end{aligned}$$

Substituting the second condition into the first condition gives the conditions

$$U_{C,1} = \beta D_K U_{C,2} D_{Y,2}\tilde{F}_{Y,K,2} \tag{3.46}$$

$$\tilde{F}_{Y,B,2} = D_K\tilde{F}_{Y,K,2}D_{Y,1}\tilde{F}_{Y,B,1}. \tag{3.47}$$

These conditions completely characterize the solution.

**Effects of climate change**

I will now vary the climate related variables to see how equilibrium fossil-fuel use reacts to these changes. Consider a change $\Delta$ that affects $M_1$, $M_2$, $D_K$, $D_{Y,1}$ and $D_{Y,2}$. The equilibrium conditions (3.46) and (3.47) can be differentiated with respect to $\Delta$ to identify the effect on the equilibrium choice of investment and allocation of fossil-fuel use. I will let primes denote full derivatives with respect to $\Delta$.

The derivatives of the production function are

$$\begin{aligned} Y_1' &= Y_1 \left[ \frac{D_{Y,1}'}{D_{Y,1}} + \frac{\tilde{F}_{Y,B,1}}{\tilde{F}_{Y,1}} B_1' \right] \\ Y_2' &= Y_2 \left[ \frac{D_{Y,2}'}{D_{Y,2}} + \frac{\tilde{F}_{Y,B,2} B_2' + \tilde{F}_{Y,K,2} K_2'}{\tilde{F}_{Y,2}} \right]. \end{aligned}$$

Differentiating both sides of condition (3.46) gives

$$\begin{aligned} \frac{d}{d\Delta} U_{C,1} &= U_{C,1} \left[ \frac{U_{CC,1}}{U_{C,1}} \left( Y_1' - \frac{K_2'}{D_K} + \frac{K_2}{D_K} \frac{D_K'}{D_K} \right) + \frac{U_{CM,1}}{U_{C,1}} M_1' \right] = \{(3.41)\} \\ &= U_{C,1} \left[ -\frac{\theta}{C_1} \left( Y_1 \left[ \frac{D_{Y,1}'}{D_{Y,1}} + \frac{\tilde{F}_{Y,B,1}}{\tilde{F}_{Y,1}} B_1' \right] - \frac{K_2'}{D_K} + \frac{K_2}{D_K} \frac{D_K'}{D_K} \right) \right] \\ &\quad + U_{C,1} \frac{U_{CM,1}}{U_{C,1}} M_1' \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\Delta} \beta D_K U_{C,2} D_{Y,2} \tilde{F}_{Y,K,2} &= \beta D_K U_{C,2} D_{Y,2} \tilde{F}_{Y,K,2} \left[ \frac{U_{CC,2}}{U_{C,2}} Y_2' + \frac{U_{CM,2}}{U_{C,2}} M_2' \right] \\ &\quad + \beta D_K U_{C,2} D_{Y,2} \tilde{F}_{Y,K,2} \left[ \frac{D_K'}{D_K} + \frac{D_{Y,2}'}{D_{Y,2}} \right] \\ &\quad + \beta D_K U_{C,2} D_{Y,2} \tilde{F}_{Y,K,2} \left[ \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} K_2' + \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}} B_2' \right] \end{aligned}$$

Using $\beta D_K U_{C,2} D_{Y,2} \tilde{F}_{Y,K,2} = U_{C,1}$, $C_2 = Y_2$, $B_2' = -B_1'$ and (3.41) delivers

$$\begin{aligned}
\frac{d}{d\Delta}\beta D_K U_{C,2} D_{Y,2}\tilde{F}_{Y,K,2} = & U_{C,1}\left[-\theta\left(\frac{D'_{Y,2}}{D_{Y,2}} - \frac{\tilde{F}_{Y,B,2}}{\tilde{F}_{Y,2}}B'_1 + \frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}}K'_2\right)\right] \\
& + U_{C,1}\left[\frac{U_{CM,2}}{U_{C,2}}M'_2 + \frac{D'_K}{D_K} + \frac{D'_{Y,2}}{D_{Y,2}}\right] \\
& + U_{C,1}\left[\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}K'_2 - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}B'_1\right] \\
= & U_{C,1}\left[(1-\theta)\frac{D'_{Y,2}}{D_{Y,2}} + \frac{D'_K}{D_K} + \frac{U_{CM,2}}{U_{C,2}}M'_2\right] \\
& + U_{C,1}\left(\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} - \theta\frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}}\right)K'_2 \\
& + U_{C,1}\left(\theta\frac{\tilde{F}_{Y,B,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}\right)B'_1.
\end{aligned}$$

Setting

$$\frac{d}{d\Delta}U_{C,1} = \frac{d}{d\Delta}\beta D_K U_{C,2} D_{Y,2}\tilde{F}_{Y,K,2}$$

and rewriting produces

$$\begin{aligned}
K'_2 = & \frac{\frac{\theta}{C_1}Y_1\frac{D'_{Y,1}}{D_{Y,1}} + \left(1+\frac{\theta}{C_1}\frac{K_2}{D_K}\right)\frac{D'_K}{D_K} + (1-\theta)\frac{D'_{Y,2}}{D_{Y,2}} - \frac{U_{CM,1}}{U_{C,1}}M'_1 + \frac{U_{CM,2}}{U_{C,2}}M'_2}{\frac{\theta}{C_1}\frac{1}{D_K} + \theta\frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}} \\
& + \frac{\left(\frac{\theta}{C_1}Y_1\frac{\tilde{F}_{Y,B,1}}{\tilde{F}_{Y,1}} + \theta\frac{\tilde{F}_{Y,B,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}\right)}{\frac{\theta}{C_1}\frac{1}{D_K} + \theta\frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}B'_1.
\end{aligned} \tag{3.48}$$

Turning, instead, to the Hotelling rule (3.47), we have that

$$\begin{aligned}
\frac{d\tilde{F}_{Y,B,2}}{d\Delta} = & \tilde{F}_{Y,B,2}\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}K'_2 + \frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}}B'_2\right) \\
= & \tilde{F}_{Y,B,2}\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}K'_2 - \frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}}B'_1\right)
\end{aligned}$$

and that

$$\begin{aligned}\frac{d}{d\Delta} D_K \tilde{F}_{Y,K,2} D_{Y,1} \tilde{F}_{Y,B,1} &= D_K \tilde{F}_{Y,K,2} D_{Y,1} \tilde{F}_{Y,B,1} \left[ \frac{D_K'}{D_K} + \frac{D_{Y,1}'}{D_{Y,1}} + \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} K_2' \right] \\ &\quad + D_K \tilde{F}_{Y,K,2} D_{Y,1} \tilde{F}_{Y,B,1} \left[ \frac{\tilde{F}_{Y,KB,2}}{\tilde{F}_{Y,K,2}} B_2' + \frac{\tilde{F}_{Y,BB,1}}{\tilde{F}_{Y,B,1}} B_1' \right] \\ &= \tilde{F}_{Y,B,2} \left[ \frac{D_K'}{D_K} + \frac{D_{Y,1}'}{D_{Y,1}} + \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} K_2' \right] \\ &\quad + \tilde{F}_{Y,B,2} \left[ \frac{\tilde{F}_{Y,BB,1}}{\tilde{F}_{Y,B,1}} - \frac{\tilde{F}_{Y,KB,2}}{\tilde{F}_{Y,K,2}} \right] B_1'.\end{aligned}$$

Setting

$$\frac{d\tilde{F}_{Y,B,2}}{d\Delta} = \frac{d}{d\Delta} D_K \tilde{F}_{Y,K,2} D_{Y,1} \tilde{F}_{Y,B,1},$$

using $B_2' = -B_1'$ and rearranging then leads to

$$K_2' = \frac{\frac{\tilde{F}_{Y,BB,1}}{\tilde{F}_{Y,B,1}} + \frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}} B_1' + \frac{\frac{D_K'}{D_K} + \frac{D_{Y,1}'}{D_{Y,1}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}.$$

Substituting this into (3.48) and rearranging gives that

$$\xi_B B_1' = \xi_{D_K} \frac{D_K'}{D_K} + \xi_{D_{Y,1}} \frac{D_{Y,1}'}{D_{Y,1}} + (\theta - 1) \frac{D_{Y,2}'}{D_{Y,2}} + \frac{U_{CM,1}}{U_{C,1}} M_1' - \frac{U_{CM,2}}{U_{C,2}} M_2', \quad (3.49)$$

where

$$\xi_{D_K} = \frac{\theta \frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} + \frac{\theta}{C_1} \left( \frac{1}{K_2} - \left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) \right) \frac{K_2}{D_K}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}} \quad (3.50)$$

$$\xi_{D_{Y,1}} = \frac{\theta \frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} + \frac{\theta}{C_1} \left[ \frac{1}{D_K} - Y_1 \left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) \right]}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}. \quad (3.51)$$

I show in appendix 3.B.2 that, for both of the two cases considered below (exogenous and endogenous supply of alternative energy),

$$\xi_B > 0 \quad (3.52)$$

By determining the signs of $\xi_{D_K}$ and $\xi_{D_{Y,1}}$, I can use (3.49) to determine the signs of the effects on first-period fossil-fuel use of varying $D_K$, $D_{Y,1}$, $D_{Y,2}$, $M_1$ and $M_2$. I will do this for two different cases, with exogenous and endogenous $S$ respectively.

### Alternative energy exogenously given

I will now assume that the amount of alternative energy in each period, $S$, is exogenously given. Under this assumption, capital is only used in final goods production and there are no indirect effects related to the redistribution of resources between final goods production and production of alternative energy. Then the derivatives of final goods production with respect to capital and fossil-fuel use are

$$\begin{aligned}
\tilde{F}_{Y,B} &= F_{Y,E}F_{E,B} > 0\\
\tilde{F}_{Y,K} &= F_{Y,G}G_K > 0\\
\tilde{F}_{Y,BK} &= F_{Y,EG}F_{E,B}G_K > 0\\
\tilde{F}_{Y,BB} &= F_{Y,EE}\left(F_{E,B}\right)^2 + F_{Y,E}F_{E,BB} < 0\\
\tilde{F}_{Y,KK} &= F_{Y,GG}\left(G_K\right)^2 + F_{Y,G}G_{KK} < 0.
\end{aligned}$$

In appendix 3.B.2 I show that these derivatives imply that $\xi_B > 0$.

Turning to $\xi_{D_K}$ and $\xi_{D_{Y,1}}$, it can be seen in (3.50) and (3.51) that they contain the following three expressions that can be calculated using the derivatives of $\tilde{F}$ and the derivatives of CES-production functions (see appendix 3.A):

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} = \theta\frac{F_{Y,G}G_K}{F_Y} - \frac{F_{Y,EG}G_K}{F_{Y,E}} = \left(\theta - \frac{1}{\sigma_Y}\right)\frac{F_{Y,G}G_K}{F_Y} \tag{3.53}$$

$$\begin{aligned}
\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \theta\frac{F_{Y,G}G_K}{F_Y} - \frac{F_{Y,GG}G_K^2 + F_{Y,G}G_{KK}}{F_{Y,G}G_K}\\
&= \theta\frac{F_{Y,G}G}{F_Y}\alpha\frac{1}{K} + \frac{1}{\sigma_Y}\frac{F_{Y,E}}{F_Y}\frac{E}{G}\alpha\frac{G}{K} + (1-\alpha)\frac{1}{K}\\
&= \left[\alpha\frac{(\theta-1)F_{Y,G}G + \frac{1-\sigma_Y}{\sigma_Y}F_{Y,E}E}{F_Y} + 1\right]\frac{1}{K}
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \frac{F_{Y,EG}G_K}{F_{Y,E}} - \frac{F_{Y,GG}G_K}{F_{Y,K}} - \frac{G_{KK}}{G_K} = \frac{1}{\sigma_Y}\frac{G_K}{G} - \frac{G_{KK}}{G_K}\\
&= \frac{1}{\sigma_Y}\alpha\frac{1}{K} - (\alpha-1)\frac{1}{K} = \left(\alpha\frac{1-\sigma_Y}{\sigma_Y} + 1\right)\frac{1}{K}
\end{aligned} \tag{3.55}$$

Substituting (3.53) and (3.55) into (3.50) delivers

$$\xi_{D_K} = -\frac{\left(\frac{1}{\sigma_Y} - \theta\right)\frac{F_{Y,G,2}G_{K,2}}{F_{Y,2}} + \frac{\theta}{C_1}\alpha\frac{1-\sigma_Y}{\sigma_Y}\frac{1}{D_K}}{\left(\alpha\frac{1-\sigma_Y}{\sigma_Y} + 1\right)\frac{1}{K_2}}. \tag{3.56}$$

Substituting (3.54) and (3.55) into (3.51) gives (after some calculations)

$$\xi_{D_{Y,1}} = -\frac{\frac{\theta}{C_1}\left(Y_1 - \frac{K_2}{D_K}\right) - 1 + \alpha\left(\frac{1}{\sigma_Y} - \theta\right)\frac{F_{Y,G,2}G_2}{F_{Y,2}} + \alpha\left(\theta\frac{Y_1}{C_1} - 1\right)\frac{1-\sigma_Y}{\sigma_Y}}{\alpha\frac{1-\sigma_Y}{\sigma_Y} + 1} \tag{3.57}$$

I can now state the following proposition:

**Proposition 3.2.** *Assume that the parameters fulfill* (3.45) *and that* $U_{CM} \geq 0$. *Then*

$$\frac{\partial B_1}{\partial D_K} \leq 0,\ \frac{\partial B_1}{\partial D_{Y,2}} \geq 0,\ \frac{\partial B_1}{\partial M_1} \geq 0,\ \frac{\partial B_1}{\partial M_2} \leq 0$$

*If, also,* $\delta = 1$, *then*

$$\frac{\partial B_1}{\partial D_{Y,1}} \leq 0$$

*Proof.* Starting from (3.49) and using that $\xi_B > 0$, the results for $D_{Y,2}$, $M_1$ and $M_2$ follow from the parameter assumptions in (3.45) and $U_{CM} \geq 0$. The result for $D_K$ follows from (3.56) and the parameter assumptions in (3.45). When $\delta = 1$, $Y_1 = C_1 + \frac{K_2}{D_K}$. This implies that $\frac{Y_1}{C_1} > 1$ and that $\frac{\theta}{C_1}\left(Y_1 - \frac{K_2}{D_K}\right) - 1 = \theta - 1$. The result for $D_{Y,1}$ now follows from (3.57) and the parameter assumptions in (3.45). □

The changes described in the proposition can be understood intuitively. If $D_K$ decreases, second-period production and consumption levels go down and more consumption is redistributed to the second period. This redistribution can be achieved either by saving on fossil fuel or by investing in more capital. The decrease in $D_K$ decreases second-period capital, which increases the marginal product of capital but decreases the marginal product of fossil fuel. Therefore it is better to use more fossil fuel in the first period which allows for investing more in capital. So more fossil fuel is used in the first period to increase production. Investment also rises (and it should go up enough to decrease first-period consumption).

A decrease in $D_{Y,2}$ decreases the second-period levels of production and consumption. This leads to an increase in second-period marginal utility from consumption. At the same time, the marginal products of both fossil fuel and capital in the second period decrease due to decreased productivity. If $\theta \geq 1$, the effect on the marginal utility of consumption dominates the effect on the marginal products and both the first-period level of investment and second-period fossil-fuel use increase. This leads to a decrease in first-period fossil-fuel use.

An increase in $M_2$ (assuming $U_{CM} \geq 0$) increase the marginal value of consumption in the second period relative to the first period. This leads to a redistribution of resources from the first to the second period. This means that both first-period investment and second-period fossil-fuel use increases. This leads to a decrease in first-period fossil-fuel use. An increase in $M_1$ works the other way around.

A decrease in $D_{Y,1}$ increases the marginal utility of consumption and decreases the marginal product of fossil fuel in the first period. At the same time first-period investments decrease, giving less second-period capital. This increases second-period marginal utility from consumption and decreases second-period marginal product of fossil fuel relative to the second-period marginal product of capital. If $\delta = 1$, the entire net income in period 1 comes from period 1 production and then more fossil fuel will be used in the first period to increase first-period consumption and investment. If, instead, $\delta < 1$, a share of first-period net income comes from non-depreciated capital. This weakens the effect on first-period consumption and second-period capital while the effect on first-period marginal product of fossil fuel is the same. This means that it is no longer certain that the best thing is to compensate decreased productivity in the first period with more fossil-fuel use.

If climate change is interpreted as a decrease in $D_K$ and $D_{Y,2}$ and an increase in $M_2$, then proposition 3.2 states that damages affecting productivity and utility move emissions from the first period to the second period while damages affecting depreciation move emissions from the second period to the first period.

**Alternative energy endogenously determined**

I will now assume that alternative-energy generation is endogenously determined. Generating alternative energy requires the use of labor and capital. I will assume that the production function for alternative energy is such that the generated amount of energy is linear in the use of the composite $G$, defined in (3.42). I will also assume that the division of $G$ into final good production and alternative-energy generation is a static decision that can be made in each period. Let $G_Y$ and $G_S$ be the amounts of $G$ used in final goods production and alternative-energy generation, respectively. The amount of alternative energy is then

$$S = A_S G_S.$$

In the previous section, with alternative energy exogenously given, the change in fossil-fuel use caused by a change in "climate" was driven by two different aspects. The first aspect was the redistribution of consumption between the periods. If the relative value of consumption increases in one period the equilibrium changes in such a way that it redistributes consumption towards that period. The second aspect was the relative productivity of the two factors ($B$ and $K$) within the periods. The first aspect will be similar in this situation. The second aspect will be different here. This is because the substitutability between the two factors is different when the alternative energy can be adjusted. In the production function $F$, defined in (3.44), the substitutability between $G$ and $E$ is low. With exogenously given alternative energy, this translates into low substitutability between $B$ and $K$ in $\tilde{F}$. In this section, the possibility to use $G$ to generate energy, increases the substitutability between $B$ and $K$ in $\tilde{F}$.

When considering changes in $D_{Y,2}$, $M_1$ and $M_2$ in the previous section, they were driven by consumption reallocation and the effects of varying these will therefore be the same here as there. This can also be seen in (3.49), where the factors in front of the changes in these climate variables do not depend on the production function $\tilde{F}$ but rather on the shape of the utility function.

When considering changes in $D_{Y,1}$ and $D_K$, the effects depend on the shape of the production function $\tilde{F}$. The factors multiplying them in (3.49), that is (3.50) and (3.51), depend on derivatives of $\tilde{F}$. In particular, they depend the following three combinations of derivatives:

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} \tag{3.58}$$

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} \tag{3.59}$$

$$\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} \tag{3.60}$$

The reallocation of $G$ between final-good production and alternative-energy generation must be taken into account when differentiating $\tilde{F}$ with respect to $B$ and $K$. I will first solve the problem of intratemporal allocation of $G$. This will allow me to calculate the needed derivatives of $\tilde{F}$ with respect to $B$ and $K$.

Becuase the division of $G$ between final goods production and alternative-energy generation is a static decision that can be made in each period, $G_Y$ and $G_S$ can be seen as functions of $B$ and $G$. The division of $G$ is thus made to maximize production in the period. Production is given by

$$D_Y F_Y\left(G_Y, F_E\left(B, A_S G_S\right)\right).$$

The intratemporal problem of dividing $G$ can then be written

$$\max_{G_Y,G_S} D_Y F_Y\left(G_Y, F_E\left(B, A_S G_S\right)\right) \text{ s.t. } G_Y + G_S = G$$

The first-order conditions for this problem give the optimality condition

$$F_{Y,G} = F_{Y,E} F_{E,S} A_S. \tag{3.61}$$

This condition implicitly defines $G_Y$ and $G_S$ as functions of $B$ and $G$. This condition can be differentiated to derive the partial derivatives of $G_Y$ and $G_S$ with respect to $B$ and $G$. This yields (see appendix 3.B.1, page 139)

$$G_{S,G} = \frac{\frac{1}{\sigma_Y}\frac{1}{G_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}} \in (0,1).$$

This expression reveals that as $G$ is increased, $G_S$ is increased. Since this derivative lies between zero and one, this implies that $G_Y$ also increases.

The production function $\tilde{F}$ can now be written

$$\tilde{F}(B,K) = F_Y\left(G_Y(B,G(K)), F_E\left(B, A_S G_S(B,G(K))\right)\right). \tag{3.62}$$

This expression can be differentiated to give the partial derivatives (see appendix 3.B.1, page 140)

$$\begin{aligned}
\tilde{F}_{Y,B} &= F_{Y,E} F_{E,B} \\
\tilde{F}_{Y,K} &= F_{Y,G} G_K \\
\tilde{F}_{Y,BK} &= F_{Y,EG} G_K F_{E,B} - F_{Y,G}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_{E,B}}{E} G_{S,G} G_K \\
\tilde{F}_{Y,KK} &= F_{Y,GG} G_K^2 + F_{Y,G} G_{KK} + \frac{1}{\sigma_Y} F_{Y,G} \frac{1}{G_Y} G_{S,G} G_K^2.
\end{aligned}$$

The first-order derivatives are the same here as in the case with exogenous alternative energy. This is because the envelope theorem implies that, since the allocation of $G$ into $G_Y$ and $G_S$ is chosen to maximize production, the reallocation gives no first-order effects on $\tilde{F}$. The second-order derivatives consists of the same terms as in the case with exogenous alternative energy. In addition to these, each derivative has a term, of opposite sign, that captures the reallocation of $G$.

These derivatives can now be used to calculate expressions (3.58) and (3.60). The underlying calculations can be found in appendix 3.B.1 (see page 143); here I only state some of the results.

Starting with expression (3.58) we can write

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} = \left(\theta - \frac{1}{\sigma_Y}\right)\frac{F_{Y,G}}{F_Y} G_K + \left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_Y}{F_{Y,E}E} G_{S,G}\frac{F_{Y,G}}{F_Y} G_K.$$

Comparing this equation to (3.53), we see that the first terms are the same. The second term captures the redistribution of $G$ that results from a change in $G$. The first term is negative and the second term is positive, so that the new element here will tend to change the sign of the expression.

Substituting for $G_{S,G}$ and rewriting delivers the following equation (see appendix 3.B.1, page 143)

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y}-\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}=\frac{\left(\theta-\frac{1}{\sigma_E}\right)\frac{1}{\sigma_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K$$
$$+\frac{\left(\theta\left(\frac{1}{\sigma_Y}\frac{F_{E,S}S}{F_E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\right)-\frac{1}{\sigma_E\sigma_Y}\right)\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K.$$

This expression has the same sign as

$$\left(\theta-\frac{1}{\sigma_E}\right)\frac{1}{\sigma_Y}+\left(\theta\left(\frac{1}{\sigma_Y}\frac{F_{E,S}S}{F_E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\right)-\frac{1}{\sigma_E\sigma_Y}\right)\frac{G_Y}{G_S} \tag{3.63}$$

The first term here is positive. The sign of the second term is ambiguous. It will be positive if $F_{E,S}S >> F_{E,B}B$ and negative if $F_{E,S}S << F_{E,B}B$. In order for (3.63) to be negative, $F_{E,B}B$ would have to be large enough compared to $F_{E,S}S$ and $G_Y$ must be large enough relative to $G_S$. Given that the energy sources are good substitutes, the income share of fossil fuel $F_{E,B}B$ will be large compared to the income share of alternative energy, $F_{E,S}S$ when a large amount of fossil fuel is used. Since energy and other inputs are poor substitutes, much of $G$ will be used in final good production if there is a significant amount of energy. This implies that (3.63) will tend to be negative if fossil fuel is abundant and positive otherwise.

Turning instead to expression (3.60), we see that it can be written (see appendix 3.B.1, page 143) as

$$\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}-\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}}=\frac{1}{\sigma_Y}\frac{1}{G_Y}G_K-\frac{G_{KK}}{G_K}$$
$$-\left[\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}-\frac{1}{\sigma_E}\frac{F_{Y,G}G_Y}{F_{Y,E}E}\right]\frac{1}{G_Y}G_{S,G}G_K.$$

Here the first two terms are the same as in the previous section, see (3.55), and they are positive. The third term, captures the effects of redistribution of $G$. As for expression just discussed, this term has the opposite sign, that is, it is negative. So, again, the effect of redistribution of $G$ will tend to change the sign of the effect compared to the previous section.

Substituting for $G_{S,G}$ and rewriting gives (see appendix 3.B.1, page 143)

$$\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}-\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}}=\alpha\frac{\frac{1}{\sigma_Y}\frac{1-\sigma_E}{\sigma_E}+\left(\frac{1}{\sigma_Y}\frac{1-\sigma_E}{\sigma_E}\frac{F_{E,S}S}{F_E}+\frac{1}{\sigma_E}\frac{1-\sigma_Y}{\sigma_Y}\frac{F_{E,B}B}{F_E}\right)\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{1}{K}+\frac{1}{K}.$$

In the expression for $\xi_{D_K}$, this shows up as

$$1-\left(\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}-\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}}\right)K.$$

This expression will have the same sign as

$$\frac{1}{\sigma_Y}\frac{\sigma_E-1}{\sigma_E}+\left(\frac{1}{\sigma_Y}\frac{\sigma_E-1}{\sigma_E}\frac{F_{E,S}S}{F_E}+\frac{1}{\sigma_E}\frac{\sigma_Y-1}{\sigma_Y}\frac{F_{E,B}B}{F_E}\right)\frac{G_Y}{G_S}. \tag{3.64}$$

The first term here is positive. The sign of the second term depends on the relative sizes of $F_{E,S}S$ and $F_{E,B}B$. It is positive if $F_{E,S}S >> F_{E,B}B$, in which case (3.64) will be positive. If $F_{E,B}B >> F_{E,S}S$, the second term will be negative and the sign of (3.64) will depend on the relative sizes of the terms. The second term will be large if $F_{E,B}B$ is large relative to $F_{E,S}S$ and if $G_Y$ is large relative to $G_S$. Thus, the analysis is the same here as for expression (3.63). This means that $\xi_{D_K}$ will be positive if $F_{E,B,2}B_2$ is large compared to $F_{E,S,2}S_2$ and $G_{Y,2}$ is large compared to $G_{S,2}$.

Turning now to $\xi_{D_{Y,1}}$ (see (3.51)), its sign depends on (3.59) and

$$\begin{aligned}\left[\frac{1}{D_K}-Y_1\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\right)\right]&=\left[\frac{\frac{K_2}{D_K}}{Y_1}-\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\right)K_2\right]\frac{Y_1}{K_2}\\&=\left[\frac{\frac{K_2}{D_K}-Y_1}{Y_1}+1-\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\right)K_2\right]\frac{Y_1}{K_2}.\end{aligned}$$

Apart from the first term in the parenthesis, this is the same expression as for $\xi_{D_K}$. The first term can be either positive or negative. If there is full depreciation it is negative since the numerator then is investment minus production, which must be negative. Expression (3.59) is positive since $\tilde{F}_{Y,K} > 0$ and $\tilde{F}_{Y,KK} < 0$.

The conclusion from this discussion is that the effect of changes in $D_{Y,2}$, $M_1$ and $M_2$ will be the same here as in the case with exogenously given alternative energy. The effect of changes in $D_K$ and $D_{Y,1}$, however, are ambiguous here.

### 3.3.2 The coal case

I will now instead consider the coal case. Here, there is no scarcity of fossil fuel, by assumption; fossil fuel can be extracted using labor and capital. I will assume that the composite input $G = K^{\alpha}L^{1-\alpha}$ is used in final goods production, fossil-fuel extraction and alternative-energy generation. A more general formulation would be to specify different production functions for different sectors describing how labor and capital are used together. Assuming that capital and labor enters the production function in the same way in all sectors simplifies the analysis but should not, if the differences are not too large, affect the results significantly.

The composite input, $G = K^{\alpha}L^{1-\alpha}$, can be used in final goods production, extraction of fossil fuel or production of alternative energy. This allocation is made in each period and is (assuming free mobility of resources between sectors between time periods) a static decision.

I assume that both the extraction of fossil fuel and the alternative-energy generation are linear in $G$, so that

$$F_B(G_B) = A_B G_B \text{ and } F_S(G_S) = A_S G_S. \tag{3.65}$$

Under these assumptions, the planner solution can be divided into two separate subproblems. The first subproblem is the within-period allocation of a given amount of $G$ between the different sectors. The second subproblem is the intertemporal problem of how much, out of first-period production, to invest into second-period capital. I will solve the two subproblems in turn.

#### Within-period allocation of $G$

I will now solve the problem of allocating a given amount of $G$ in order to maximize production. Let the amount of $G$ used in final goods production, fossil-fuel extraction and alternative-energy generation be labeled $G_Y$, $G_B$ and $G_S$, respectively. The problem of allocating $G$ optimally is then

$$\max_{G_Y, G_B, G_S} F_Y\left(G_Y, F_E\left(F_B\left(G_B\right), F_S\left(G_S\right)\right)\right) \text{ s.t. } G_B + G_S + G_Y = G.$$

There should also be non-negativity constraints on each of the $G$s; however, given the parameter assumptions in (3.45), these constraints will never bind.

The Lagrangian of this problem is

$$\mathcal{L} = F_Y\left(G_Y, F_E\left(F_B\left(G_B\right), F_S\left(G_S\right)\right)\right) + \lambda\left[G - G_B - G_S - G_Y\right].$$

The first-order conditions are

$$\begin{aligned} G_B : \lambda &= F_{Y,E} F_{E,B} F_{B,G} \\ G_S : \lambda &= F_{Y,E} F_{E,S} F_{S,G} \\ G_Y : \lambda &= F_{Y,G}. \end{aligned}$$

These first-order conditions express that the resulting marginal final good product in all sectors should be the same. They can be rewritten to give the equilibrium conditions

$$F_{E,S} F_{S,G} = F_{E,B} F_{B,G} \tag{3.66}$$

$$F_{Y,G} = F_{Y,E} F_{E,B} F_{B,G}. \tag{3.67}$$

For a given $G$, these conditions implicitly determine the allocation of $G$ between different uses.

Using the production functions (3.43) and (3.65) and the derivatives of CES-production functions from appendix 3.A, condition (3.66) now delivers

$$\begin{aligned} A_S \gamma_S \left(\tfrac{F_E}{A_S G_S}\right)^{\frac{1}{\sigma_E}} &= A_B \gamma_B \left(\tfrac{F_E}{A_B G_B}\right)^{\frac{1}{\sigma_E}} \\ \Rightarrow G_S &= \left(\tfrac{A_S}{A_B}\right)^{\sigma_E - 1} \left(\tfrac{\gamma_S}{\gamma_B}\right)^{\sigma_E} G_B. \end{aligned}$$

Let

$$\xi_S = \left(\frac{A_S}{A_B}\right)^{\sigma_E - 1} \left(\frac{\gamma_S}{\gamma_B}\right)^{\sigma_E} > 0 \Rightarrow G_S = \xi_S G_B.$$

Substituting this expression into the production function for the energy composite, we obtain

$$E = \left[\gamma_B B^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S S^{\frac{\sigma_E - 1}{\sigma_E}}\right] = \left[\gamma_B A_B^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S \left(A_S \xi_S\right)^{\frac{\sigma_E - 1}{\sigma_E}}\right] G_B.$$

Similarly, let

$$\xi_E = \left[\gamma_B A_B^{\frac{\sigma_E - 1}{\sigma_E}} + \gamma_S \left(A_S \xi_S\right)^{\frac{\sigma_E - 1}{\sigma_E}}\right] > 0 \Rightarrow E = \xi_E G_B.$$

Condition (3.67) then gives

$$\begin{aligned} \gamma_G \left(\frac{F_Y}{G_Y}\right)^{\frac{1}{\sigma_Y}} &= \gamma_E \left(\frac{F_Y}{E}\right)^{\frac{1}{\sigma_Y}} \gamma_B \left(\frac{F_E}{B}\right)^{\frac{1}{\sigma_E}} A_B \\ &= \gamma_E \left(\frac{F_Y}{\xi_E G_B}\right)^{\frac{1}{\sigma_Y}} \gamma_B \left(\frac{\xi_E G_B}{A_B G_B}\right)^{\frac{1}{\sigma_E}} A_B \\ &= \gamma_E \gamma_B \left(\frac{F_Y}{G_B}\right)^{\frac{1}{\sigma_Y}} \xi_E^{\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}} A_B^{\frac{\sigma_E}{\sigma_E}}. \end{aligned}$$

Finally, let

$$\xi_Y = \left[\frac{\gamma_G}{\gamma_E\gamma_B}\xi_E^{\frac{1}{\sigma_Y}-\frac{1}{\sigma_E}} A_B^{\frac{1-\sigma_E}{\sigma_E}}\right]^{\sigma_Y} > 0 \Rightarrow G_Y = \xi_Y G_B$$

Summing up the different uses of the composite $G$ one then finds that

$$G = G_Y + G_B + G_S = (1+\xi_Y+\xi_S)G_B \Rightarrow G_B = \frac{1}{1+\xi_Y+\xi_S}G$$

This in turn delivers

$$G_Y = \frac{\xi_Y}{1+\xi_Y+\xi_S}G,\ G_B = \frac{1}{1+\xi_Y+\xi_S}G \text{ and } G_S = \frac{\xi_S}{1+\xi_Y+\xi_S}G. \tag{3.68}$$

Thus, the composite $G$ is divided between the different uses in proportions that are independent of $G$ and also independent of $D_{Y,1}$, $D_{Y,2}$, $D_K$, $M_1$ and $M_2$. This also implies that fossil-fuel use is increasing in $G$.

Final goods production can now be written

$$Y = A_Y\left[\gamma_G G_Y^{\frac{\sigma_Y-1}{\sigma_Y}} + \gamma_E E^{\frac{\sigma_Y-1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y-1}} = A_Y\left[\gamma_G \xi_Y^{\frac{\sigma_Y-1}{\sigma_Y}} + \gamma_E \xi_E^{\frac{\sigma_Y-1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y-1}} G_B$$
$$= A_Y\left[\gamma_G \xi_Y^{\frac{\sigma_Y-1}{\sigma_Y}} + \gamma_E \xi_E^{\frac{\sigma_Y-1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y-1}} \frac{1}{1+\xi_Y+\xi_S}G.$$

Let

$$\xi_{YG} = \left[\gamma_G \xi_Y^{\frac{\sigma_Y-1}{\sigma_Y}} + \gamma_E \xi_E^{\frac{\sigma_Y-1}{\sigma_Y}}\right]^{\frac{\sigma_Y}{\sigma_Y-1}} \frac{1}{1+\xi_Y+\xi_S},$$

giving

$$Y = A_Y\xi_{YG}G. \tag{3.69}$$

For the results derived below, there are two intratemporal implications that are particularly interesting. The first of these is that, as can be seen in (3.68), the amount of fossil fuel extracted in a period is strictly increasing in $G$ and therefore also in the amount of capital. The second result is that production in a period can be written as a factor, which is independent of $G$, $D_{Y,1}$, $D_{Y,2}$, $D_K$, $M_1$ and $M_2$, times $G$. This means that production in a period, as a function of capital, inherits many of the properties of $G$ as a function of $K$. In particular,

$$\frac{\partial Y}{\partial K} > 0 \text{ and } \frac{\partial^2 Y}{\partial K^2} < 0. \tag{3.70}$$

### Intertemporal allocation

Since the allocation of $G$ between the different sectors is a static decision, that can be made in each period, production in each period can be written as a function of only capital

$$Y_t = D_{Y,t}\hat{F}(K_t).$$

The planner problem is therefore

$$\max_{K_2}\begin{bmatrix} U\left(D_{Y,1}\hat{F}\left(K_1\right)+\left(1-\delta\right)K_1-\frac{K_2}{D_K},M_1\right) \\ +\beta U\left(D_{Y,2}\hat{F}\left(K_2\right),M_2\right)\end{bmatrix}$$

The first-order condition with respect to $K_2$ can be stated as

$$U_{C,1} = \beta D_K U_{C,2} D_{Y,2}\hat{F}'(K_2).$$

Consider now a change $\Delta$ that affects $D_K$, $D_{Y,1}$, $D_{Y,2}$, $M_1$ and $M_2$. Let primes denote full derivatives with respect to $\Delta$. Then

$$\begin{aligned}
\frac{dU_{C,1}}{d\Delta} &= U_{CC,1}\left[Y_1\frac{D'_{Y,1}}{D_{Y,1}}-\frac{K'_2}{D_K}+\frac{K_2}{D_K}\frac{D'_K}{D_K}\right]+U_{CM,1}M'_1 \\
\frac{d}{d\Delta}\beta D_K U_{C,2}D_{Y,2}\hat{F}'(K_2) &= U_{C,1}\left[\frac{U_{CC,2}}{U_{C,2}}\left(Y_2\frac{D'_{Y,2}}{D_{Y,2}}+D_{Y,2}\hat{F}'(K_2)K'_2\right)\right] \\
&\quad +U_{C,1}\left[\frac{D'_K}{D_K}+\frac{D'_{Y,2}}{D_{Y,2}}+\frac{\hat{F}''(K_2)}{\hat{F}'(K_2)}K'_2\right] \\
&\quad +U_{C,1}\frac{U_{CM,2}}{U_{C,2}}M'_2 \\
&= \left\{\frac{U_{CC}}{U_C}=-\frac{\theta}{C}\text{ and } C_2=Y_2\right\}= \\
&= U_{C,1}\left[(1-\theta)\frac{D'_{Y,2}}{D_{Y,2}}+\frac{D'_K}{D_K}+\frac{U_{CM,2}}{U_{C,2}}M'_2\right] \\
&\quad +U_{C,1}\left[\frac{\hat{F}''(K_2)}{\hat{F}'(K_2)}-\theta\frac{\hat{F}'(K_2)}{\hat{F}(K_2)}\right]K'_2.
\end{aligned}$$

Equating these derivatives and rearranging delivers

$$\begin{aligned}
K'_2 &= \frac{\frac{\theta}{C_1}Y_1\frac{D'_{Y,1}}{D_{Y,1}}+(1-\theta)\frac{D'_{Y,2}}{D_{Y,2}}+\left(1+\frac{\theta}{C_1}\frac{K_2}{D_K}\right)\frac{D'_K}{D_K}}{\frac{\theta}{C_1}\frac{1}{D_K}+\theta\frac{\hat{F}'(K_2)}{\hat{F}(K_2)}-\frac{\hat{F}''(K_2)}{\hat{F}'(K_2)}} \\
&\quad +\frac{-\frac{U_{CM,1}}{U_{C,1}}M'_1+\frac{U_{CM,2}}{U_{C,2}}M'_2}{\frac{\theta}{C_1}\frac{1}{D_K}+\theta\frac{\hat{F}'(K_2)}{\hat{F}(K_2)}-\frac{\hat{F}''(K_2)}{\hat{F}'(K_2)}}
\end{aligned} \tag{3.71}$$

The following proposition can now be stated:

**Proposition 3.3.** *In the coal case, the first-period fossil-fuel use is exogenously determined. The second-period fossil-fuel use depends on* $D_{Y,1}$*,* $D_{Y,2}$*,* $D_K$*,* $M_1$ *and* $M_2$ *as follows*

$$\frac{\partial B_2}{\partial D_{Y,1}} \geq 0,\ \frac{\partial B_2}{\partial D_{Y,2}} \leq 0,\ \frac{\partial B_2}{\partial D_K} \geq 0,\ \frac{\partial B_2}{\partial M_1} \leq 0 \text{ and } \frac{\partial B_2}{\partial M_2} \geq 0$$

*Proof.* First-period capital is given. Since this means that $G_1$ is given, equation (3.68) implies that first-period fossil-fuel use is given. For second-period fossil-fuel use, (3.68) gives that $\frac{dB_2}{dK_2} > 0$. From (3.70) it follows that $\hat{F}' > 0$ and that $\hat{F}'' < 0$, since they have the same signs as $\frac{\partial Y}{\partial K}$ and $\frac{\partial^2 Y}{\partial K^2}$, respectively. The results then follow from equation (3.71) since the denominator on the right-hand side is positive, $\theta \geq 1$ and $U_{CM} \geq 0$. □

The results can be interpreted as follows. First-period capital, and therefore also fossil-fuel use, is given. If $D_{Y,1}$ decreases, production in the first period and the amount of resources that can be divided between first-period consumption and investment into second-period capital both decrease. This leads to decreases in both $C_1$ and $K_2$. The decrease in $K_2$ implies a decrease in $B_2$. If, instead, $D_{Y,2}$ decreases, then the marginal product of second-period capital must decrease while the marginal utility of second-period consumption increases. Which of these effects dominates depends on the sign of $\theta - 1$. If $\theta > 1$, the effect on marginal utility from consumption in the second period dominates. First-period investment then increases giving increased $G_2$ and $B_2$.

If $D_K$ decreases, there is a direct effect of a decrease in second-period capital. There is also a decrease in the amount of second-period capital that each unit of savings gives. Counteracting this, the value of second-period capital increases both due to increased marginal product of capital and the increased marginal utility from consumption. So there may be an increase in first-period investment, but this effect will not be large enough to counteract the effects of increased depreciation. Thus, a decrease in $D_K$ leads to a decrease in second-period capital which implies a decrease in second-period fossil-fuel use.

Changes in $M_1$ and $M_2$ do not affect the production possibilities. Instead, they affect the relative value of consumption between the two periods, thus inducing a redistribution of consumption between the two periods. This occurs by changing first-period investment. If the value of first-period consumption decreases relative to second-period consumption, first-period investment increases. This increases the amount of second-period capital which implies an increase in second-period fossil-fuel use. If the value of first-period consumption increases relative to second-period consumption, the opposite will occur.

### 3.3.3 Amplification or dampening of climate change

Sections 3.3.1 and 3.3.2 derived the results of how exogenous changes affected the equilibrium allocation. The intended interpretation is that these changes are in fact driven by climate change which is driven by fossil-fuel use and, therefore, endogenous. In this section, I will discuss how, in a model with endogenous climate, the change in the allocation due to the effects of climate change would affect fossil-fuel use. In particular I will discuss whether the reactions to the realization that there is climate change might amplify or dampen climate change.

To begin with, how will climate change manifest itself in the model?

Climate change will decrease $D_Y$ and $D_K$ and increase $M$. Since $D_K$ gives the amount of depreciation between the periods, the effects of climate change can be interpreted as a decrease in $D_K$. For changes in $D_Y$ and $M$ I have to determine if they primarily affect the period 1 or period 2 values. As has been seen above, the timing of the changes matters crucially. Thus, for changes in $D_Y$ and $M$ I must determine what the effects will be in the different periods. On the one hand, climate change is a slow process; one the other hand, the use of a two period model implies that a time period should be considered as being relatively long. For the purpose of the present discussion, I will make the assumption that the effects on the second-period values are larger than the effects on the first-period values.

To say whether the reactions to climate change will dampen or amplify the change, I must also determine what I mean by amplification or dampening. These interpretations will be somewhat different in the oil and coal cases. In the oil case, total fossil-fuel use is exogenously given by the total initial supply. Thus, only its timing is endogenously determined. I will then interpret a reallocation of fossil-fuel use from the second to the first period as an amplification of climate change. In the coal case, first-period fossil-fuel use is determined by first-period capital. Second-period fossil-fuel use is endogenously determined by the first-period choice of second-period capital. In that case, I will interpret an increase in second-period fossil-fuel use as an amplification of climate change.

Based on the assumption about what the effects of climate change will be, and the interpretation of what amplification of climate change means, propositions 3.2 and 3.3 give the effects in the different cases. Proposition 3.2 says that in the oil case, with exogenously given alternative energy, climate effects on productivity or utility will dampen climate change while climate effects on depreciation will amplify climate change. Proposition 3.3 says that in the coal case, climate effects on productivity or utility will amplify climate change while climate effects on depreciation will dampen climate change. Thus, we conclude that climate effects on depreciation will have the opposite effect compared to climate effects affecting productivity or utility directly.

Furthermore, these effects will be reversed in the coal case compared to the oil case: the dampening or amplifying effects of climate change, for a given type of damages, are the opposite in the oil and coal case.[3] Still, the effects are similar in both cases in terms of in which direction the intertemporal change goes. The difference is largely driven by what is exogenously given. In the oil case total fossil-fuel use is given while in the coal case, first-period fossil-fuel use is given.

## 3.4 Discussion

This chapter has investigated how the nature of climate externalities influences our analysis of the effects of climate change on the economy. I looked at damages to productivity, capital depreciation and utility directly and I carried out the analysis in two complementary ways.

Firstly, I derived a formula that, under some specific assumptions, gave the optimal per unit fossil-fuel tax. The formula is very simple and states that the optimal tax in any period, in relation to total production in that period, equals a specific constant. This constant is a sum of three different parts, one for each type of climate effect. This formula serves as an aggregating device: it allows us to combine different types of effects of climate change into one measure.

The formula is very appealing in its simplicity. Its derivation required specific assumptions for some aspects of the model, while other aspects remained very general. Perhaps the strongest assumptions are those regarding the shape of the damage functions. While these assumptions may be reasonable for moderate climate change, they cannot capture the risks of catastrophic climate change.

Secondly,using a two-period model, I considered how the different possible effects of climate change affect fossil-fuel use. I considered two different cases. In the oil case, fossil fuel is costlessly extracted from a given total supply. In that case, all fossil fuel is always used. In the coal case, the extraction of fossil fuel requires inputs, but the scarcity of the fossil fuel does not matter. For the oil case, I considered one case where alternative energy was exogenously given, and one case where alternative

[3] At least if considering the oil case with exogenously given alternative energy

energy was endogenously determined by the amounts of inputs used in the generation of it. The main conclusion from this analysis is that the qualitative results depend crucially on the assumptions made. Thus, the assumptions made need to be carefully motivated.

In the two-period model, I treated climate as exogenous. I do not consider this assumption very problematic. As discussed, the endogenous responses may either dampen or amplify climate change, but it does not seem likely that they would overturn the qualitative results regarding the signs of the changes in fossil-fuel use.

Restricting the analysis to a two-period model means that it is only possible to have changes between the first and the second periods. In a multi-period model, more complicated patterns of changes are possible. My conjecture is that the qualitative results derived here would generalize to a multi-period model where the climate changes monotonically. Since the climate is expected to deteriorate for quite a long time, the restriction to a two-period model need therefore not be so restrictive.

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## 3.A Derivatives of a CES production function

The production function is

$$F(G,E) = \left[\gamma_G G^{\frac{\sigma-1}{\sigma}} + \gamma_E E^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

This yields that

$$\begin{aligned}
F_G &= \gamma_G \left(\frac{F}{G}\right)^{\frac{1}{\sigma}} \\
F_{GG} &= F_G\left(\frac{1}{\sigma}\frac{F_G}{F} - \frac{1}{\sigma}\frac{1}{G}\right) = -\frac{1}{\sigma}\frac{F_G}{G}\gamma_E\left(\frac{E}{F}\right)^{\frac{\sigma-1}{\sigma}} = -\frac{1}{\sigma}\frac{F_G F_E}{F}\frac{E}{G} \\
F_E &= \gamma_E \left(\frac{F}{E}\right)^{\frac{1}{\sigma}} \\
F_{EE} &= F_E\left(\frac{1}{\sigma}\frac{F_E}{F} - \frac{1}{\sigma}\frac{1}{E}\right) = -\frac{1}{\sigma}\frac{F_E}{E}\gamma_G\left(\frac{G}{F}\right)^{\frac{\sigma-1}{\sigma}} = -\frac{1}{\sigma}\frac{F_G F_E}{F}\frac{G}{E} \\
F_{EG} &= F_{GE} = \frac{1}{\sigma}\frac{F_G F_E}{F}.
\end{aligned}$$

The following combinations of derivatives are utilized in the derivations:

$$\begin{aligned}
\frac{F_{GG}}{F_G} - \theta\frac{F_G}{F} &= -\frac{1}{G}\frac{\frac{1}{\sigma}F_E E + \theta F_G G}{F} \\
\theta\frac{F_E}{F} - \frac{F_{EG}}{F_G} &= \left(\theta - \frac{1}{\sigma}\right)\frac{F_E}{F} \\
\frac{F_{EG}}{F_E} - \frac{F_{GG}}{F_G} &= \frac{1}{\sigma}\frac{1}{G} \\
\frac{F_{EG}}{F_G} - \frac{F_{EE}}{F_E} &= \frac{1}{\sigma}\frac{1}{E} \\
F_{EE}F_{GG} = F_{EG}^2 &= \frac{1}{\sigma^2}\frac{F_G^2 F_E^2}{F^2}.
\end{aligned}$$

## 3.B Calculations for the oil case

### 3.B.1 Calculations for the oil case with endogenous alternative energy

**Calculation of $G_{Y,B}$ and $G_{S,B}$**

When differentiating with respect to $B$, total $G$ is fixed. This means that $G_{Y,B} = -G_{S,B}$.

Differentiating both sides of (3.61) with respect to $B$ delivers

$$\begin{aligned}
\frac{dF_{Y,G}}{dB} &= F_{Y,G}\left[\frac{F_{Y,GG}}{F_{Y,G}}G_{Y,B} + \frac{F_{Y,GE}}{F_{Y,G}}\left(F_{E,B} + F_{E,S}A_S G_{S,B}\right)\right] \\
&= F_{Y,G}\left[\frac{F_{Y,GE}}{F_{Y,G}}F_{E,B} + \left(\frac{F_{Y,GE}}{F_{Y,G}}F_{E,S}A_S - \frac{F_{Y,GG}}{F_{Y,G}}\right)G_{S,B}\right] \\
&= \{(3.61)\} = F_{Y,G}\left[\frac{F_{Y,GE}}{F_{Y,G}}F_{E,B} + \left(\frac{F_{Y,GE}}{F_{Y,E}} - \frac{F_{Y,GG}}{F_{Y,G}}\right)G_{S,B}\right] \\
&= F_{Y,G}\left[\frac{F_{Y,GE}}{F_{Y,G}}F_{E,B} + \frac{1}{\sigma_Y}\frac{1}{G_Y}G_{S,B}\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{dF_{Y,E}F_{E,S}A_S}{dB} &= F_{Y,E}F_{E,S}A_S\left[\frac{F_{Y,GE}}{F_{Y,E}}G_{Y,B} + \frac{F_{Y,EE}}{F_{Y,E}}\left(F_{E,B} + F_{E,S}A_S G_{S,B}\right)\right] \\
&\quad + F_{Y,E}F_{E,S}A_S\left[\frac{F_{E,BS}}{F_{E,S}} + \frac{F_{E,SS}}{F_{E,S}}A_S G_{S,B}\right] = \{(3.61)\} = \\
&= F_{Y,G}\left[\left(\frac{F_{Y,EE}}{F_{Y,E}}\frac{F_{Y,G}}{F_{Y,E}} + \frac{F_{E,SS}}{F_{E,S}}A_S - \frac{F_{Y,GE}}{F_{Y,E}}\right)G_{S,B}\right] \\
&\quad + F_{Y,G}\left[\frac{F_{Y,EE}}{F_{Y,E}}F_{E,B} + \frac{F_{E,BS}}{F_{E,S}}\right] \\
&= F_{Y,G}\left[\left(\left(\frac{F_{Y,EE}}{F_{Y,E}} - \frac{F_{Y,GE}}{F_{Y,E}}\right)\frac{F_{Y,G}}{F_{Y,E}} + \frac{F_{E,SS}}{F_{E,S}}A_S\right)G_{S,B}\right] \\
&\quad + F_{Y,G}\left[\frac{F_{Y,EE}}{F_{Y,E}}F_{E,B} + \frac{F_{E,BS}}{F_{E,S}}\right] \\
&= F_{Y,G}\left[\left(\frac{F_{E,SS}}{F_{E,S}}A_S - \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}}\right)G_{S,B}\right] \\
&\quad + F_{Y,G}\left[\frac{F_{Y,EE}}{F_{Y,E}}F_{E,B} + \frac{F_{E,BS}}{F_{E,S}}\right].
\end{aligned}$$

Equating these derivatives and solving for $G_{S,B}$ produces

$$
\begin{aligned}
G_{S,B} = -G_{Y,B} &= \frac{\frac{F_{Y,EE}}{F_{Y,E}} F_{E,B} + \frac{F_{E,BS}}{F_{E,S}} - \frac{F_{Y,EG}}{F_{Y,G}} F_{E,B}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}} A_S} \\
&= \frac{\left(\frac{F_{Y,EE}}{F_{Y,E}} - \frac{F_{Y,EG}}{F_{Y,G}}\right) F_{E,B} + \frac{F_{E,BS}}{F_{E,S}}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}} A_S} \\
&= \frac{-\frac{1}{\sigma_Y}\frac{1}{E} F_{E,B} + \frac{1}{\sigma_E}\frac{F_{E,B}}{F_E}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}} A_S} = \frac{\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{E} F_{E,B}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}} A_S} \\
&= \left\{ -\frac{F_{E,SS}}{F_{E,S}} = \frac{1}{\sigma_E}\frac{F_{E,B}}{F_E}\frac{B}{S} A_S = \frac{1}{\sigma_E}\frac{F_{E,B} B}{F_E}\frac{1}{G_S} \right\} = \\
&= \frac{\left(\frac{1}{\sigma_E} - \frac{1}{\sigma_Y}\right)\frac{1}{E} F_{E,B}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E} E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B} B}{F_E}\frac{1}{G_S}}.
\end{aligned}
\tag{3.72}
$$

**Calculation of $G_{Y,G}$ and $G_{S,G}$**

When differentiating with respect to $G$, $G_{Y,G} + G_{S,G} = 1$. Differentiating both sides of (3.61) with respect to $G$ yields

$$
\begin{aligned}
\frac{dF_{Y,G}}{dG} &= F_{Y,G}\left[\frac{F_{Y,GG}}{F_{Y,G}} G_{Y,G} + \frac{F_{Y,GE}}{F_{Y,G}} F_{E,S} A_S G_{S,G}\right] \\
&= F_{Y,G}\left[\frac{F_{Y,GG}}{F_{Y,G}} + \left(\frac{F_{Y,GE}}{F_{Y,G}} F_{E,S} A_S - \frac{F_{Y,GG}}{F_{Y,G}}\right) G_{S,G}\right] \\
&= \{(3.61)\} = F_{Y,G}\left[\frac{F_{Y,GG}}{F_{Y,G}} + \left(\frac{F_{Y,GE}}{F_{Y,E}} - \frac{F_{Y,GG}}{F_{Y,G}}\right) G_{S,G}\right] \\
&= F_{Y,G}\left[\frac{F_{Y,GG}}{F_{Y,G}} + \frac{1}{\sigma_Y}\frac{1}{G_Y} G_{S,G}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dF_{Y,E}F_{E,S}A_S}{dG} &= F_{Y,E}F_{E,S}A_S\left[\frac{F_{Y,GE}}{F_{Y,E}}G_{Y,G} + \frac{F_{Y,EE}}{F_{Y,E}}F_{E,S}A_S G_{S,G}\right] \\
&\quad + F_{Y,E}F_{E,S}A_S\frac{F_{E,SS}}{F_{E,S}}A_S G_{S,G} = \{(3.61)\} = \\
&= F_{Y,G}\left(\frac{F_{Y,EE}}{F_{Y,E}}F_{E,S}A_S + \frac{F_{E,SS}}{F_{E,S}}A_S - \frac{F_{Y,GE}}{F_{Y,E}}\right)G_{S,G} \\
&\quad + F_{Y,G}\frac{F_{Y,GE}}{F_{Y,E}} = \{(3.61)\} = \\
&= F_{Y,G}\left[\left(\frac{F_{Y,EE}}{F_{Y,E}} - \frac{F_{Y,GE}}{F_{Y,E}}\right)\frac{F_{Y,G}}{F_{Y,E}} + \frac{F_{E,SS}}{F_{E,S}}A_S\right]G_{S,G} \\
&\quad + F_{Y,G}\frac{F_{Y,GE}}{F_{Y,E}} \\
&= F_{Y,G}\left[\left(-\frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} + \frac{F_{E,SS}}{F_{E,S}}A_S\right)G_{S,G} + \frac{F_{Y,GE}}{F_{Y,E}}\right].
\end{aligned}
$$

Equating these derivatives and solving for $G_{S,G}$ leads to

$$
\begin{aligned}
G_{S,G} &= \frac{\frac{F_{Y,GE}}{F_{Y,E}} - \frac{F_{Y,GG}}{F_{Y,G}}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}}A_S} \\
&= \left\{-\frac{F_{E,SS}}{F_{E,S}} = \frac{1}{\sigma_E}\frac{F_{E,B}}{F_E}\frac{B}{S}A_S = \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}\right\} = \\
&= \frac{\frac{1}{\sigma_Y}\frac{1}{G_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}} \in (0,1)
\end{aligned}
\tag{3.73}
$$

and

$$
G_{Y,G} = 1 - G_{S,G} = \frac{\frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}}A_S}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}} \in (0,1).
$$

**Calculation of derivatives of $\tilde{F}(B,K)$**

Differentiating (3.62) delivers

$$
\begin{aligned}
\tilde{F}_{Y,B} &= F_{Y,G}G_{Y,B} + F_{Y,E}\left(F_{E,B} + F_{E,S}A_S G_{S,B}\right) \\
&= \left(F_{Y,G} - F_{Y,E}F_{E,S}A_S\right)G_{Y,B} + F_{Y,E}F_{E,B} \\
&= \{(3.61)\} = F_{Y,E}F_{E,B} \\
\tilde{F}_{Y,K} &= F_{Y,G}G_{Y,G}G_K + F_{Y,E}F_{E,S}A_S G_{S,G}G_K \\
&= F_{Y,G}G_K + \left(F_{Y,E}F_{E,S}A_S - F_{Y,G}\right)G_{S,G}G_K \\
&= \{(3.61)\} = F_{Y,G}G_K.
\end{aligned}
$$

Differentiating $\tilde{F}_{Y,B}$ gives

$$\begin{aligned}
\tilde{F}_{Y,BB} &= \left(F_{Y,EG}G_{Y,B} + F_{Y,EE}\left(F_{E,B} + F_{E,S}A_S G_{S,B}\right)\right)F_{E,B} \\
&\quad + F_{Y,E}\left(F_{E,BB} + F_{E,BS}A_S G_{S,B}\right) = \{(3.61)\} = \\
&= F_{Y,EE}F_{E,B}^2 + F_{Y,E}F_{E,BB} \\
&\quad + \left(F_{Y,G}\left(\frac{F_{Y,EG}}{F_{Y,G}} - \frac{F_{Y,EE}}{F_{Y,E}}\right)F_{E,B} - F_{Y,E}F_{E,BS}A_S\right)G_{Y,B} \\
&= F_{Y,EE}F_{E,B}^2 + F_{Y,E}F_{E,BB} + F_{Y,G}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_{E,B}}{F_E}G_{Y,B} \\
\tilde{F}_{Y,BK} &= \left(F_{Y,EG}G_{Y,G}G_K + F_{Y,EE}F_{E,S}A_S G_{S,G}G_K\right)F_{E,B} \\
&\quad + F_{Y,E}F_{E,BS}A_S G_{S,G}G_K = \{(3.61)\} = \\
&= F_{Y,EG}G_K F_{E,B} \\
&\quad + \left[\left(F_{Y,EE}\frac{F_{Y,G}}{F_{Y,E}} - F_{Y,EG}\right)F_{E,B} + F_{Y,E}F_{E,BS}A_S\right]G_{S,G}G_K \\
&= F_{Y,EG}G_K F_{E,B} \\
&\quad + F_{Y,G}\left[\left(\frac{F_{Y,EE}}{F_{Y,E}} - \frac{F_{Y,EG}}{F_{Y,G}}\right)F_{E,B} + \frac{F_{E,BS}}{F_{E,S}}\right]G_{S,G}G_K \\
&= F_{Y,EG}G_K F_{E,B} - F_{Y,G}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_{E,B}}{E}G_{S,G}G_K.
\end{aligned}$$

Differentiating $\tilde{F}_{Y,K}$ produces

$$\begin{aligned}
\tilde{F}_{Y,KK} &= \left[F_{Y,GG}G_{Y,G}G_K + F_{Y,EG}F_{E,S}A_S G_{S,G}G_K\right]G_K + F_{Y,G}G_{KK} \\
&= \left[F_{Y,GG}\left(1 - G_{S,G}\right) + F_{Y,EG}\frac{F_{Y,G}}{F_{Y,E}}G_{S,G}\right]G_K^2 + F_{Y,G}G_{KK} \\
&= F_{Y,GG}G_K^2 + F_{Y,G}G_{KK} + F_{Y,G}\left(\frac{F_{Y,EG}}{F_{Y,E}} - \frac{F_{Y,GG}}{F_{Y,G}}\right)G_{S,G}G_K^2 \\
&= F_{Y,GG}G_K^2 + F_{Y,G}G_{KK} + \frac{1}{\sigma_Y}F_{Y,G}\frac{1}{G_Y}G_{S,G}G_K^2.
\end{aligned}$$

Substituting for $G_{Y,B}$ from (3.72) in $\tilde{F}_{Y,BB}$ and rewriting yields

$$\begin{aligned}\tilde{F}_{Y,BB} =& F_{Y,EE}F_{E,B}^2 + F_{Y,E}F_{E,BB} \\ &+ \left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right) F_{Y,G}F_{E,B}\frac{1}{E}\frac{\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{1}{E}F_{E,B}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}} \\ =& \left\{\frac{F_{Y,G}}{F_{Y,EE}} = -\frac{F_{Y,G}}{\frac{1}{\sigma_Y}\frac{F_{Y,G}F_{Y,E}}{F_Y}\frac{G_Y}{E}} = -\frac{F_Y}{\frac{1}{\sigma_Y}F_{Y,E}E}\frac{1}{G_Y}E^2\right\} = \\ =& F_{Y,EE}\left(1 - \frac{\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)^2\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y}}{\frac{1}{\sigma_Y^2}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_Y\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}}\right)F_{E,B}^2 + F_{Y,E}F_{E,BB} < 0.\end{aligned}$$

The parenthesis in the last expression lies between 0 and 1 since

$$0 < \left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)^2 \frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} < \frac{1}{\sigma_Y^2}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y}.$$

Substituting $G_{S,G}$ from (3.73) into $\tilde{F}_{Y,BK}$ implies

$$\begin{aligned}\tilde{F}_{Y,BK} =& F_{Y,EG}G_KF_{E,B} \\ &-F_{Y,G}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_{E,B}}{E}\frac{\frac{1}{\sigma_Y}\frac{1}{G_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}}G_K \\ =& \left\{\frac{F_{Y,G}}{F_{Y,EG}} = \frac{F_{Y,G}}{\frac{1}{\sigma_Y}\frac{F_{Y,E}F_{Y,G}}{F_Y}} = \frac{\frac{F_Y}{F_{Y,E}}}{\frac{1}{\sigma_Y}}\right\} = \\ =& F_{Y,EG}\left(1 - \frac{\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{1}{G_S}}\right)F_{E,B}G_K > 0.\end{aligned}$$

The parenthesis in the last expression lies between 0 and 1 since

$$0 < \frac{1}{\sigma_Y}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y} < \frac{1}{\sigma_Y^2}\frac{F_Y}{F_{Y,E}E}\frac{1}{G_Y}.$$

Substituting for $G_{S,G}$ in $\tilde{F}_{Y,KK}$ gives

$$\begin{aligned}
\tilde{F}_{Y,KK} &= F_{Y,GG} G_K^2 + F_{Y,G} G_{KK} \\
&\quad + \frac{1}{\sigma_Y} F_{Y,G} \frac{1}{G_Y} \frac{\frac{1}{\sigma_Y} \frac{1}{G_Y}}{\frac{1}{\sigma_Y} \frac{F_Y}{F_{Y,E} E} \frac{1}{G_Y} + \frac{1}{\sigma_E} \frac{F_{E,B} B}{F_E} \frac{1}{G_S}} \\
&= \left\{ \frac{\frac{1}{\sigma_Y} F_{Y,G} \frac{1}{G_Y}}{F_{Y,GG}} = \frac{\frac{1}{\sigma_Y} F_{Y,G} \frac{1}{G_Y}}{-\frac{1}{\sigma_Y} \frac{F_{Y,G} F_{Y,E}}{F_Y} \frac{E}{G_Y}} = -\frac{F_Y}{F_{Y,E} E} \right\} = \\
&= F_{Y,GG} \left( 1 - \frac{\frac{1}{\sigma_Y} \frac{F_Y}{F_{Y,E} E} \frac{1}{G_Y}}{\frac{1}{\sigma_Y} \frac{F_Y}{F_{Y,E} E} \frac{1}{G_Y} + \frac{1}{\sigma_E} \frac{F_{E,B} B}{F_E} \frac{1}{G_S}} \right) G_K^2 + F_{Y,G} G_{KK} < 0.
\end{aligned}$$

The parenthesis in the last expression lies between 0 and 1.

**Calculating expressions** (3.58) **and** (3.59)

Using the derivatives of $\tilde{F}$ from the previous section, we obtain

$$\begin{aligned}
\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} &= \frac{F_{Y,G}}{F_Y} G_K \\
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} &= \frac{F_{Y,EG} G_K F_{E,B} - F_{Y,G} \left( \frac{1}{\sigma_Y} - \frac{1}{\sigma_E} \right) \frac{F_{E,B}}{E} G_{S,G} G_K}{F_{Y,E} F_{E,B}} \\
&= \frac{F_{Y,EG}}{F_{Y,E}} G_K - \left( \frac{1}{\sigma_Y} - \frac{1}{\sigma_E} \right) \frac{F_{Y,G}}{F_{Y,E} E} G_{S,G} G_K \\
\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \frac{F_{Y,GG} G_K^2 + F_{Y,G} G_{KK} + F_{Y,G} \frac{1}{\sigma_Y} \frac{1}{G_Y} G_{S,G} G_K^2}{F_{Y,G} G_K} \\
&= \frac{F_{Y,GG}}{F_{Y,G}} G_K + \frac{G_{KK}}{G_K} + \frac{1}{\sigma_Y} \frac{1}{G_Y} G_{S,G} G_K.
\end{aligned}$$

These can be combined so that we can express (3.58) as follows:

$$\begin{aligned}
\theta \frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} &= \theta \frac{F_{Y,G}}{F_Y} G_K - \frac{F_{Y,EG}}{F_{Y,E}} G_K + \left( \frac{1}{\sigma_Y} - \frac{1}{\sigma_E} \right) \frac{F_{Y,G}}{F_{Y,E} E} G_{S,G} G_K \\
&= \left[ \left( \theta - \frac{1}{\sigma_Y} \right) \frac{F_{Y,G}}{F_Y} + \left( \frac{1}{\sigma_Y} - \frac{1}{\sigma_E} \right) \frac{F_{Y,G}}{F_{Y,E} E} G_{S,G} \right] G_K \\
&= \left[ \theta - \frac{1}{\sigma_Y} + \left( \frac{1}{\sigma_Y} - \frac{1}{\sigma_E} \right) \frac{F_Y}{F_{Y,E} E} G_{S,G} \right] \frac{F_{Y,G}}{F_Y} G_K.
\end{aligned}$$

$G_{S,G}$ from (3.73) can be rewritten as

$$G_{S,G} = \frac{\frac{1}{\sigma_Y}\frac{1}{G_Y}}{\frac{1}{\sigma_Y}\frac{1}{G_Y} + \frac{1}{\sigma_Y}\frac{1}{E}\frac{F_{Y,G}}{F_{Y,E}} - \frac{F_{E,SS}}{F_{E,S}}A_S} = \frac{\frac{1}{\sigma_Y}}{\frac{1}{\sigma_Y} + \frac{1}{\sigma_Y}\frac{F_{Y,G}G_Y}{F_{Y,E}E} - \frac{F_{E,SS}}{F_{E,S}}A_S G_Y}$$
$$= \frac{\frac{1}{\sigma_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E S}A_S G_Y} = \frac{\frac{1}{\sigma_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}.$$

Using this, we obtain

$$\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y} - \frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} = \left[\theta - \frac{1}{\sigma_Y} + \frac{\frac{1}{\sigma_Y}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_Y}{F_{Y,E}E}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\right]\frac{F_{Y,G}}{F_Y}G_K$$
$$= \frac{\left(\theta - \frac{1}{\sigma_Y}\right)\left(\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}\right)}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K$$
$$+ \frac{\frac{1}{\sigma_Y}\left(\frac{1}{\sigma_Y} - \frac{1}{\sigma_E}\right)\frac{F_Y}{F_{Y,E}E}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K$$
$$= \frac{\left(\theta - \frac{1}{\sigma_E}\right)\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K$$
$$+ \frac{\left(\theta - \frac{1}{\sigma_Y}\right)\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K.$$

Furthermore, the condition for the allocation of $G$ between the sectors, equation (3.61), implies that

$$\frac{F_Y}{F_{Y,E}E} = \frac{F_{Y,E}E + F_{Y,G}G_Y}{F_{Y,E}E} = 1 + \frac{F_{Y,G}G_Y}{F_{Y,E}E} = \{(3.61)\} = 1 + F_{E,S}A_S\frac{G_Y}{E}$$
$$= \{S = A_S G_S\} = 1 + \frac{F_{E,S}S}{F_E}\frac{G_Y}{G_S}.$$

Using this equation, we see that

$$\begin{aligned}
\theta\frac{\tilde{F}_{Y,K}}{\tilde{F}_Y}-\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} &= \frac{\left(\theta-\frac{1}{\sigma_E}\right)\frac{1}{\sigma_Y}\left(1+\frac{F_{E,S}S}{F_E}\frac{G_Y}{G_S}\right)}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K \\
&\quad+\frac{\left(\theta-\frac{1}{\sigma_Y}\right)\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K \\
&= \left\{\frac{F_{E,B}B}{F_E}+\frac{F_{E,S}S}{F_E}=1\right\} \\
&= \frac{\left(\theta-\frac{1}{\sigma_E}\right)\frac{1}{\sigma_Y}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K \\
&\quad+\frac{\left(\theta\left(\frac{1}{\sigma_Y}\frac{F_{E,S}S}{F_E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\right)-\frac{1}{\sigma_E\sigma_Y}\right)\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}+\frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{F_{Y,G}}{F_Y}G_K.
\end{aligned}$$

Similarly, calculating the expression in (3.59) instead, we arrive at

$$\begin{aligned}
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}-\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \left(\frac{F_{Y,EG}}{F_{Y,E}}-\frac{F_{Y,GG}}{F_{Y,G}}\right)G_K-\frac{G_{KK}}{G_K} \\
&\quad-\left[\left(\frac{1}{\sigma_Y}-\frac{1}{\sigma_E}\right)\frac{F_{Y,G}}{F_{Y,E}E}+\frac{1}{\sigma_Y}\frac{1}{G_Y}\right]G_{S,G}G_K \\
&= \frac{1}{\sigma_Y}\frac{1}{G_Y}G_K-\frac{G_{KK}}{G_K} \\
&\quad-\left[\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}-\frac{1}{\sigma_E}\frac{F_{Y,G}G_Y}{F_{Y,E}E}\right]\frac{1}{G_Y}G_{S,G}G_K.
\end{aligned}$$

Using $G_K=\alpha\frac{G}{K}$, $G_{KK}=(\alpha-1)\frac{G_K}{K}$, this equation becomes

$$\begin{aligned}
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}}-\frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \frac{1}{\sigma_Y}\alpha\frac{G}{G_Y}\frac{1}{K}-(\alpha-1)\frac{1}{K} \\
&\quad-\left[\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}-\frac{1}{\sigma_E}\frac{F_{Y,G}G_Y}{F_{Y,E}E}\right]G_{S,G}\alpha\frac{G}{G_Y}\frac{1}{K} \\
&= \frac{1}{K}+\alpha\left[\frac{1}{\sigma_Y}\frac{G}{G_Y}-1\right]\frac{1}{K} \\
&\quad-\alpha\left[\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E}-\frac{1}{\sigma_E}\frac{F_{Y,G}G_Y}{F_{Y,E}E}\right]G_{S,G}\frac{G}{G_Y}\frac{1}{K}.
\end{aligned}$$

Substituting for $G_{S,G}$ delivers

$$\begin{aligned}
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \frac{1}{K} + \alpha\left[\frac{1}{\sigma_Y}\frac{G}{G_Y} - 1\right]\frac{1}{K} \\
&\quad -\alpha\frac{\frac{1}{\sigma_Y}\left[\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} - \frac{1}{\sigma_E}\frac{F_{Y,G}G_Y}{F_{Y,E}E}\right]}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{G}{G_Y}\frac{1}{K} \\
&= \left\{\text{Canceling two terms } \frac{1}{\sigma_Y^2}\frac{F_Y}{F_{Y,E}E}\right\} \\
&= \frac{1}{K} - \alpha\frac{1}{K} \\
&\quad +\alpha\frac{1}{\sigma_Y\sigma_E}\frac{\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S} + \frac{F_{Y,G}G_Y}{F_{Y,E}E}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{G}{G_Y}\frac{1}{K}.
\end{aligned}$$

Using equation (3.61), we obtain

$$\frac{F_{Y,G}G_Y}{F_{Y,E}E} = \frac{F_{E,S}S}{F_E}\frac{G_Y}{G_S} \Rightarrow \frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S} + \frac{F_{Y,G}G_Y}{F_{Y,E}E} = \frac{G_Y}{G_S}.$$

Furthermore,

$$\begin{aligned}
&\frac{F_Y}{F_{Y,E}E} = 1 + \frac{F_{E,S}S}{F_E}\frac{G_Y}{G_S} \\
&\Rightarrow \frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S} = \frac{1}{\sigma_Y} + \left(\frac{1}{\sigma_Y}\frac{F_{E,S}S}{F_E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\right)\frac{G_Y}{G_S}
\end{aligned}$$

implies

$$\begin{aligned}
\frac{\tilde{F}_{Y,BK}}{\tilde{F}_{Y,B}} - \frac{\tilde{F}_{Y,KK}}{\tilde{F}_{Y,K}} &= \alpha\frac{\frac{1}{\sigma_Y\sigma_E}\frac{G_Y}{G_S}\frac{G}{G_Y} - \left(\frac{1}{\sigma_Y} + \left(\frac{1}{\sigma_Y}\frac{F_{E,S}S}{F_E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\right)\frac{G_Y}{G_S}\right)}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{1}{K} \\
&\quad +\frac{1}{K} \\
&= \left\{\frac{G_Y}{G_S}\frac{G}{G_Y} = \frac{G}{G_S} = 1 + \frac{G_Y}{G_S}\right\} \\
&= \alpha\frac{\frac{1}{\sigma_Y}\frac{1-\sigma_E}{\sigma_E} + \left(\frac{1}{\sigma_Y}\frac{1-\sigma_E}{\sigma_E}\frac{F_{E,S}S}{F_E} + \frac{1}{\sigma_E}\frac{1-\sigma_Y}{\sigma_Y}\frac{F_{E,B}B}{F_E}\right)\frac{G_Y}{G_S}}{\frac{1}{\sigma_Y}\frac{F_Y}{F_{Y,E}E} + \frac{1}{\sigma_E}\frac{F_{E,B}B}{F_E}\frac{G_Y}{G_S}}\frac{1}{K} + \frac{1}{K}.
\end{aligned}$$

### 3.B.2 Determining the sign of $\xi_B$

From the derivation of (3.49), it can be seen that $\xi_B$ is given by

$$\xi_B = \frac{\theta}{C_1} \frac{Y_1 \frac{\tilde{F}_{Y,B,1}}{\tilde{F}_{Y,1}} \left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) - \frac{1}{D_K} \left( \frac{\tilde{F}_{Y,BB,1}}{\tilde{F}_{Y,B,1}} + \frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}} \right)}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}$$
$$+ \frac{\tilde{F}_{Y,BB,1}}{\tilde{F}_{Y,B,1}} \frac{\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} - \theta \frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}$$
$$+ \frac{\left( \theta \frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) \left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}} - \frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}} \right)}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}$$
$$+ \frac{\left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) \left( \theta \frac{\tilde{F}_{Y,B,2}}{\tilde{F}_{Y,2}} - \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}} \right)}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}} \tag{3.74}$$

#### Exogenous alternative energy

Using the derivatives of $\tilde{F}$ with exogenous alternative energy from section 3.3.1 (in the subsection starting on page 120), one can observe that the first three terms in the expression (3.74) for $\xi_B$ are positive. In the fourth term, the expression

$$-\frac{\left( \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \right) \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}} = \frac{\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}} \frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}} - \frac{\tilde{F}^2_{Y,BK,2}}{\tilde{F}_{Y,B,2}\tilde{F}_{Y,K,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}} - \frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}$$

is negative. The first part of this expression can be canceled by an identical term with opposite sign from the third term of (3.74). In order to cancel the second part, it can first be noted that using the derivatives in appendix 3.A, $F^2_{Y,EG} = F_{Y,EE}F_{Y,GG}$. Then we obtain

$$\begin{aligned}
\tilde{F}_{Y,BB}\tilde{F}_{Y,KK} - \left(\tilde{F}_{Y,BK}\right)^2 &= F_{Y,EE}(F_{E,B})^2 F_{Y,GG}(G_K)^2 \\
&\quad + F_{Y,EE}(F_{E,B})^2 F_{Y,G}G_{KK} \\
&\quad + F_{Y,E}F_{E,BB}F_{Y,GG}(G_K)^2 \\
&\quad + F_{Y,E}F_{E,BB}F_{Y,G}G_{KK} \\
&\quad - \left(F_{Y,EG}\right)^2 (F_{E,B})^2 (G_K)^2 \\
&= F_{Y,EE}(F_{E,B})^2 F_{Y,G}G_{KK} \\
&\quad + F_{Y,E}F_{E,BB}F_{Y,GG}(G_K)^2 \\
&\quad + F_{Y,E}F_{E,BB}F_{Y,G}G_{KK} > 0.
\end{aligned}$$

Using this fact, and a term from the third term of (3.74), we deduce that

$$\frac{\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}^2_{Y,BK,2}}{\tilde{F}_{Y,B,2}\tilde{F}_{Y,K,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}=\frac{\frac{\tilde{F}_{Y,KK,2}\tilde{F}_{Y,BB,2}-\tilde{F}^2_{Y,BK,2}}{\tilde{F}_{Y,B,2}\tilde{F}_{Y,K,2}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}>0.$$

These arguments can be summarized to conclude

$$\xi_B > 0$$

**Endogenous alternative energy**

In section 3.B.1 (in the part starting on page 140) it is possible to observe that also with endogenous alternative energy we have

$$\tilde{F}_{Y,BB} < 0,\ \tilde{F}_{Y,BK} > 0 \text{ and } \tilde{F}_{Y,KK} < 0.$$

This means that the first two terms in (3.74) are positive. The last two terms in (3.74) are

$$\begin{aligned}&\frac{\left(\theta\frac{\tilde{F}_{Y,K,2}}{\tilde{F}_{Y,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\right)\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}-\frac{\tilde{F}_{Y,BB,2}}{\tilde{F}_{Y,B,2}}\right)}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}\\&+\frac{\left(\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}\right)\left(\theta\frac{\tilde{F}_{Y,B,2}}{\tilde{F}_{Y,2}}-\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,K,2}}\right)}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}\\=&\frac{2\theta\frac{\tilde{F}_{Y,BK}}{\tilde{F}_Y}+\frac{\tilde{F}_{Y,BB}\tilde{F}_{Y,KK}}{\tilde{F}_{Y,B}\tilde{F}_{Y,K}}-\frac{\tilde{F}^2_{Y,BK}}{\tilde{F}_{Y,B}\tilde{F}_{Y,K}}-\theta\frac{\tilde{F}_{Y,K}\tilde{F}_{Y,BB}}{\tilde{F}_Y\tilde{F}_{Y,B}}-\theta\frac{\tilde{F}_{Y,B}\tilde{F}_{Y,KK}}{\tilde{F}_Y\tilde{F}_{Y,K}}}{\frac{\tilde{F}_{Y,BK,2}}{\tilde{F}_{Y,B,2}}-\frac{\tilde{F}_{Y,KK,2}}{\tilde{F}_{Y,K,2}}}.\end{aligned}$$

All the terms in the denominator of this expression are positive except for $-\frac{\tilde{F}^2_{Y,BK}}{\tilde{F}_{Y,B}\tilde{F}_{Y,K}}$. It can, however, be shown that

$$2\theta\frac{\tilde{F}_{Y,BK}}{\tilde{F}_Y}+\frac{\tilde{F}_{Y,BB}\tilde{F}_{Y,KK}}{\tilde{F}_{Y,B}\tilde{F}_{Y,K}}-\frac{\tilde{F}^2_{Y,BK}}{\tilde{F}_{Y,B}\tilde{F}_{Y,K}}>0.$$

This implies that $\xi_B > 0$ also in this case.

# Chapter 4

# Indirect Effects of Climate Change

## 4.1 Introduction

The direct effects of climate change are expected to differ a great deal across the world. Some countries are expected to suffer very severe consequences, where perhaps the most extreme cases are countries that are under serious risk of flooding due to sea-level rise. In contrast, some countries will only face small negative consequences of climate change, and could even benefit from moderate climate change. Apart from the differences in physical effects, the capacity to adapt to the changing conditions implied by climate change also differs a great deal across countries.

The contribution to climate change, through the emission of greenhouse gases, also varies significantly across countries. As an example, Bangladesh, a country that would be hit very hard by the effects of sea-level rise, contributes about 0.3% of world emissions, while the US contributes about 15-20% of world emissions (www.cait.org).

So both the consequences of, and the contribution to, climate change vary very much across countries. A casual look at the vulnerability to climate change and the current emissions of greenhouse gases suggests an inverse relationship between these. This poses a problem for the mitigation of climate change (reductions of emissions of greenhouse gases) since the countries that can contribute the most to mitigation efforts tend to be the countries least interested in avoiding climate change.

Measurements of vulnerability to climate change are typically based on the direct effects that will occur in a country. However, the assessment of the consequences of climate change for a country should really be based on total general equilibrium effects.

I will here consider two different types of indirect effects that will contribute to the total effects. First, I will consider trade in goods. Second, I will consider trade in financial instruments that can be used to insure against weather induced shocks.

So the first channel of indirect effects that I consider is trade in goods. A common way of modeling the damages caused by climate change is to assume that climate change affects productivity. On the world market, changes in productivity in a country will result in general equilibrium effects on the world market prices of goods. In order to capture the trade between countries that will be affected differently by climate change, I will focus on trade in different goods between countries with different comparative advantages (rather than on trade in similar goods among countries with similar comparative advantages). Therefore, I model trade using a model of the Ricardian type.

The second channel of indirect effects that I consider is trade in financial instruments that can be used to insure against variability. Climate change is expected to result in more variability in weather outcomes. It is expected to lead to an increased probability, and severity, of extreme weather events such as droughts, floods, cyclones and heatwaves (IPCC, 2007). Financial markets will allow for some insurance against these kinds of events. The prices of insuring instruments will depend on the world distribution of weather shocks. To model the trade in financial instruments, I set up a two-period endowment model. The second-period endowments are stochastic and I assume there to be complete markets to insure against this uncertainty. In this model, climate change acts by changing the second-period endowment distribution.

The general principle for the indirect effects, for both channels, is that a country benefits from changes that increase the world market price of goods or instruments of which the country is a net seller or changes that decrease the world market price of goods or instruments of which the country is a net buyer, and the other way around.

So, in general, the signs of the indirect effects will critically depend on the assumptions made about how the effects of climate change relate to the patterns of trade in goods and financial instruments.

I also apply these more general results to a two-country example. This example is intended to capture the asymmetry that can be seen between rich and poor countries. Making the asymmetry stark, I assume that emissions of greenhouse gases are only made in the rich country, while the effects of climate change are only felt in the poor country. Therefore, in this example, it would be difficult to reach agreements about climate change mitigation since the mitigation efforts must be made by the rich country that is not, directly, affected by the negative consequences of climate change. Under reasonable assumptions, I find that the channel through trade in goods will have negative indirect effects in the rich country, while the channel through trade in financial instruments will have positive indirect effects in the rich country. So the channel through trade in goods will tend to make the interests of the countries more aligned with each other, while the channel through trade in financial instruments will make the interests of the countries less aligned.

In the two-country case of trade in goods, I also consider how tariffs affect the incentives of the rich country to reduce emissions. In that case, I find that it may be possible for the poor country to induce the rich country to reduce the emissions by threatening to increase the tariffs.

Looking at the previous literature, the most widely used models for studying the interaction between the economy and the climate are the DICE and RICE models developed by Nordhaus (see e.g. Nordhaus and Boyer, 2000). These models have a homogeneous consumption good and do not have any uncertainty so the mechanisms considered here are not present at all.

Regarding climate change and trade in goods, there is a relatively large literature on the pollution haven hypothesis, in general, and the risk of carbon leakage. This refers to the risk that unilateral efforts to reduce emissions of greenhouse gases will just cause the emissions to move to other countries. Overviews of this literature are provided by Copeland and Taylor (2004) and Antweiler et al. (2001). There is also more recent work that considers how endogenous technological change affects the risk of carbon leakage (see, e.g., Di Maria and Smulders 2004; Di Maria and van der Werf, 2008; Golombek and Hoel, 2004; Hemous 2012). I will not have any carbon leakage effects in the models I use here.

Regarding insurance against climate related events, Arrow et al. (1996) and Chichilnisky (1998) both discuss some problems related to the ability of financial markets to provide insurance against climate related events. I will say more about this in section 4.4. Arrow et al. (1996) note that if there were markets for insurance against climate related events, the insurance premia would inequitably be borne by those exposed to the risks. This point is related to the way that trade in financial instruments makes the interests of the countries less aligned in this chapter.

The rest of this chapter consists of three sections. Sections 4.2 and 4.3 set up and analyze models with trade in goods and insuring financial instruments, respectively. Each of these sections starts with setting up a model with many countries and deriving results about possible changes.

These general results are then applied to the two-country case. For trade in goods, the two-country case also has a section with tariffs. Section 4.4 provides conclusions, a discussion of the results and suggestions for future work.

## 4.2 Trade in goods

In this section, I will set up a model with trade in goods between countries and use this model to analyze the total welfare effects of changes in productivities. If the productivity changes in one country, e.g., due to climate change, this will, in addition to having a direct effect on the income in that country, affect the world market prices of goods. These changes in world market prices affect the welfare in other countries.

A significant share of GDP consists of goods that are traded internationally. Part of this trade is trade in similar goods among similar countries. Another part of this trade is trade in different goods between countries with different comparative advantages. The purpose of this chapter is to capture trade between countries that are affected differently by climate change and, in the two-country case, it is intended to capture the trade between the north and the south. Therefore, the trade in different goods between countries that differ in their comparative advantages seems to be the most relevant case. The trade model used here will therefore be of the Ricardian type where countries choose what to produce based on their comparative advantages. The comparative advantages are determined by the countries' good-specific productivities. In the model that I set up here, I will assume that the comparative advantages are strong enough to induce each country to specialize in the production of one good and I will not explicitly model the choice of what to produce. In a more general model, the choice of what to produce would be endogenous based on the good-specific productivities. Each country could also produce many goods. The effects of changes in the good-specific productivities of a county, rather than the general productivity, could then be analyzed. Assuming that each country is specialized in the production of one good simplifies the analysis, but the main conclusions from the model generalize relatively straightforwardly to the more general setting, at least as long as the changes in productivities are such that the signs of net exports do not change.

I will first, in section 4.2.1, set up a many country model and, in section 4.2.2, I will solve for the equilibrium allocation. In section 4.2.3, I will then consider the welfare effects of changes in productivities. In section 4.2.4 I will look at the implications for the two-country (north and south) case. In section 4.2.5 I will consider how trading costs, in the form of import tariffs, will affect the incentives for the north to reduce the emissions of greenhouse gases. Finally, in section 4.2.6, I will briefly discuss how the indirect effects would affect the distribution of costs associated with mitigation policies.

### 4.2.1 Model setup

There are $I$ different countries and $J$ different goods with $I \geq J$. There is a measure $l_i$ of households in country $i$, each supplying one unit of labor. Production is linear in the amount of labor used. As explained above, each country, $i$, specializes in producing one good, $j_i$. Let

$$\bar{I}_j = \{i \in \{1, \dots, I\} : j_i = j\}$$

denote the set of countries that specialize in the production of good $j$ and let $a_i$ be the productivity per unit of labor in country $i$. Assume also that at least one country specializes in the production of each good, that is, $\bar{I}_j$ is non-empty for all $j$. This implies that total production of good $j$ is

$$A_j = \sum_{i \in \bar{I}_j} a_i l_i. \tag{4.1}$$

Let $c_{i,j}$ be the amount of good $j$ that is consumed by the representative household in country $i$. Preferences over consumption baskets $(c_{i,1}, \dots, c_{i,J})$ are given by

$$U(c_{i,1}, \dots, c_{i,J}) = u(c_i)$$

where

$$c_i = \left( \sum_{j=1}^{J} (c_{i,j})^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}. \tag{4.2}$$

Here, $\sigma \geq 0$ measures the substitutability between the different consumption goods and $\theta$ measures the curvature of the function giving utility from consumption of the aggregate consumption good $c_i$. Given that a good here should rather be interpreted as a relatively wide category of goods, it seems reasonable to assume that the substitutability between different goods is relatively low, that is, $\sigma$ should be relatively low.

Let $p_j$ be the price of good $j$. Markets, in each country, are assumed to be competitive and, initially, there are no trading costs. The price taking behavior that is implicit in the assumption of competitive markets is very important here since the indirect effects go through changing prices.

Competitive markets also imply that the wage is equal to the marginal value of labor, that is, the wage in country $i$ is $w_i = p_{j_i} a_i$. So, the budget constraint of the representative household in country $i$ is given by

$$a_i p_{j_i} = \sum_{j=1}^{J} p_j c_{i,j}. \tag{4.3}$$

The optimal consumption choice problem of the household is

$$\max_{c_{i,1},\ldots,c_{i,J}} U(c_{i,1},\ldots,c_{i,J}) \text{ s.t. } a_i p_{j_i} = \sum_{j=1}^{J} p_j c_{i,j}. \tag{4.4}$$

The equilibrium allocation and prices must be such that the representative household in each country $i$ maximizes utility, given prices, and such that the market clearing condition

$$A_j = \sum_{i=1}^{I} l_i c_{i,j} \tag{4.5}$$

holds for each good $j$, where $A_j$ is defined in (4.1).

### 4.2.2 Equilibrium allocation

To solve for the equilibrium allocation and prices, I will start by solving the consumption choice problem of the representative household in country $i$ and imposing market clearing. As long as the consumption goods are not perfect substitutes, $\sigma < \infty$, the household will choose an interior consumption basket such that $c_{i,j} > 0$ for all $j$. Therefore, I will not include non-negativity constraints on the quantities consumed.

The Lagrangian of the household's utility maximization problem (4.4) is

$$\mathcal{L} = U(c_{i,1},\ldots,c_{i,J}) + \lambda_i \left[ a_i p_{j_i} - \sum_{j=1}^{J} p_j c_{i,j} \right]$$

where $\lambda_i$ is a multiplier. The first-order condition with respect to $c_{i,j}$ is

$$\lambda_i p_j = \frac{\partial}{\partial c_{i,j}} u(c_i) = u'(c_i) \frac{\partial c_i}{\partial c_{i,j}} = u'(c_i) \left( \frac{c_i}{c_{i,j}} \right)^{\frac{1}{\sigma}}.$$

Comparing goods $j_1$ and $j_2$, we obtain

$$\frac{p_{j_1}}{p_{j_2}} = \left( \frac{c_{i,j_1}}{c_{i,j_2}} \right)^{-\frac{1}{\sigma}} \text{ or } c_{i,j_1} = \left( \frac{p_{j_1}}{p_{j_2}} \right)^{-\sigma} c_{i,j_2}. \tag{4.6}$$

So the relative consumption of good $j_1$ and $j_2$ depends on the relative price and the substitutability. The relative consumption levels will be the same for all countries.

Substituting (4.6) in the market clearing condition (4.5) gives that for each good $j$

$$A_j = \sum_{i=1}^{I} l_i c_{i,j} = \sum_{i=1}^{I} l_i \left(\frac{p_j}{p_{\bar{j}}}\right)^{-\sigma} c_{i,\bar{j}} = \left(\frac{p_j}{p_{\bar{j}}}\right)^{-\sigma} \sum_{i=1}^{I} l_i c_{i,\bar{j}}$$

for an arbitrary $\bar{j}$. The last sum is independent of $j$, implying that for any goods $j_1$ and $j_2$

$$\frac{p_{j_1}}{p_{j_2}} = \left(\frac{A_{j_1}}{A_{j_2}}\right)^{-\frac{1}{\sigma}}. \tag{4.7}$$

That is, the relative price depends on the relative supply and on the substitutability.

Using (4.6) in the budget constraint (4.3), for each $i$ we arrive at

$$w_i = p_{j_i} a_i = \sum_{j=1}^{J} p_j c_{i,j} = \sum_{j=1}^{J} p_j \left(\frac{p_j}{p_{\bar{j}}}\right)^{-\sigma} c_{i,\bar{j}} = c_{i,\bar{j}} p_{\bar{j}}^{\sigma} \sum_{j=1}^{J} p_j^{1-\sigma}$$

for any $\bar{j}$. Comparing representative households in country $i_1$ and $i_2$, for any $j$, it holds that

$$\frac{c_{i_1,j}}{c_{i_2,j}} = \frac{w_{i_1}}{w_{i_2}}.$$

Using the market clearing condition (4.5), for any $j$

$$A_j = \sum_{i=1}^{I} l_i c_{i,j} = \sum_{i=1}^{I} l_i c_{\bar{i},j} \frac{c_{i,j}}{c_{\bar{i},j}} = c_{\bar{i},j} \sum_{i=1}^{I} l_i \frac{w_i}{w_{\bar{i}}} = c_{\bar{i},j} \frac{\sum_{i=1}^{I} l_i w_i}{w_{\bar{i}}}.$$

This implies that for any $(i, j)$

$$c_{i,j} = A_j \frac{w_i}{\sum_{i'=1}^{I} l_{i'} w_{i'}}.$$

Define

$$s_i = \frac{w_i}{\sum_{i'=1}^{I} l_{i'} w_{i'}}. \tag{4.8}$$

This is the wealth share of a representative household in country $i$ out of total world wealth. For any $(i, j)$ we now have

$$c_{i,j} = s_i A_j.$$

Therefore, for any good $j$, the representative household in country $i$ consumes its wealth share of total production of that good. Furthermore, for any $i$

$$c_i = \left(\sum_{j=1}^{J} c_{i,j}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{j=1}^{J} (s_i A_j)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = s_i \left(\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

Let

$$A = \left( \sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{4.9}$$

be a measure of total world production, taking substitutability into account. Note the similarity between the way that the different $A_j$ are combined into $A$ and the way that the $c_{i,j}$ are combined into $c_i$ in (4.2). We thus have

$$c_i = s_i A \tag{4.10}$$

for all $i$.

Thus, we see that the equilibrium allocation is such that each household consumes its wealth share of the world production of each good and, therefore, also of total world production.

What remains now is to express the wealth share in terms of exogenous objects. In order to do this, the prices implicit in the wages must be substituted for. Using that $w_i = a_i p_{j_i}$ in (4.8) and using the relative price (4.7), the wealth share can be rewritten as

$$\begin{aligned} s_i &= \frac{w_i}{\sum_{i'=1}^{I} l_{i'} w_{i'}} = \frac{p_{j_i} a_i}{\sum_{i'=1}^{I} l_{i'} p_{j_{i'}} a_{i'}} = \frac{p_{j_i} a_i}{\sum_{j'=1}^{J} p_{j'} \sum_{i' \in \bar{I}_{j'}} l_{i'} a_{i'}} \\ &= \frac{p_{j_i} a_i}{\sum_{j'=1}^{J} p_{j'} A_{j'}} = \frac{\left( \frac{A_{j_i}}{A_{\bar{j}}} \right)^{-\frac{1}{\sigma}} p_{\bar{j}} a_i}{\sum_{j'=1}^{J} \left( \frac{A_{j'}}{A_{\bar{j}}} \right)^{-\frac{1}{\sigma}} p_{\bar{j}} A_{j'}} = \frac{A_{j_i}^{-\frac{1}{\sigma}} a_i}{\sum_{j'=1}^{J} A_{j'}^{\frac{\sigma-1}{\sigma}}}. \end{aligned}$$

This yields

$$s_i = \frac{A_{j_i}^{-\frac{1}{\sigma}} a_i}{\sum_{j'=1}^{J} A_{j'}^{\frac{\sigma-1}{\sigma}}}. \tag{4.11}$$

Since the relative prices and the allocation of consumption are now completely determined in terms of the parameters and exogenously given variables, this completes the characterization of the equilibrium. The representative household in country $i$ consumes share $s_i$, given by (4.11), of the total production of each good $j$. The relative prices are given by (4.7). One degree of freedom remains in the price determination. This is because it is only the relative prices, and not the price level, that matter. The prices can be completely determined by, for example, choosing one good as the numeraire or by defining a price index that is normalized.

### 4.2.3 Welfare effects of changes in the productivities

We can now turn to looking at the welfare effects of changes in productivities. The implied interpretation of these changes is that they are the result of climate change. In section 4.2.6, I will briefly discuss the implications of the derived results have for the division of the costs of climate change mitigation.

Since the utility of the representative household in country $i$ is strictly increasing in $c_i$, utility will change in the same direction as $c_i$. Starting from (4.10), the effect of a change in productivity $a_{\bar{i}}$ on $c_i$ is

$$\frac{dc_i}{da_{\bar{i}}} = s_i \frac{dA}{da_{\bar{i}}} + \frac{ds_i}{da_{\bar{i}}} A.$$

The first effect can be seen as a size-of-the-pie effect and the second effect as a share-of-the-pie effect.

Differentiating (4.9), we obtain

$$\frac{dA}{da_{\bar{i}}} = \frac{\sigma}{\sigma - 1} \left( \sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \frac{\sigma - 1}{\sigma} A_{j_{\bar{i}}}^{-\frac{1}{\sigma}} l_{\bar{i}} = A \frac{A_{j_{\bar{i}}}^{-\frac{1}{\sigma}} l_{\bar{i}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} > 0.$$

So, as expected, the size effect is positive for an increase in productivity in any country.

Differentiating (4.11) delivers

$$\frac{ds_i}{da_{\bar{i}}} = s_i \left( \frac{\frac{d}{da_{\bar{i}}} A_{j_i}^{-\frac{1}{\sigma}} a_i}{A_{j_i}^{-\frac{1}{\sigma}} a_i} - \frac{\frac{d}{da_{\bar{i}}} \sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \right).$$

From (4.1), the first derivative is

$$\frac{d}{da_{\bar{i}}} A_{j_i}^{-\frac{1}{\sigma}} a_i = \begin{cases} 0 & \text{if } j_{\bar{i}} \neq j_i \\ -\frac{1}{\sigma} A_{j_i}^{-\frac{1}{\sigma}-1} l_{\bar{i}} a_i & \text{if } j_{\bar{i}} = j_i \text{ but } \bar{i} \neq i \\ A_{j_i}^{-\frac{1}{\sigma}} \left(1 - \frac{1}{\sigma} \frac{l_i a_i}{A_{j_i}}\right) & \text{if } \bar{i} = i \end{cases}$$

and the second derivative is

$$\frac{d}{da_{\bar{i}}} \sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}} = \frac{\sigma - 1}{\sigma} A_{j_{\bar{i}}}^{-\frac{1}{\sigma}} l_{\bar{i}}.$$

Combining these derivatives gives the total change in the wealth share of the representative household in country $i$

$$\frac{ds_i}{da_{\bar{i}}} = \begin{cases} \frac{s_i \frac{1-\sigma}{\sigma} A_{j_{\bar{i}}}^{-\frac{1}{\sigma}} l_{\bar{i}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} & \text{if } j_{\bar{i}} \neq j_i \\ -\frac{s_i l_{\bar{i}}}{A_{j_i}} \left( 1 + \frac{1}{\sigma} \left( 1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_{j_i}^{\frac{\sigma-1}{\sigma}}} \right) \right) & \text{if } j_{\bar{i}} = j_i \text{ but } \bar{i} \neq i \\ \frac{s_i l_i}{A_{j_i}} \left( \frac{A_{j_i}}{l_i a_i} - 1 + \frac{\sigma-1}{\sigma} \left( 1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \right) \right) & \text{if } \bar{i} = i \end{cases}$$

Intuitively, the change in the wealth share can be divided into two different effects. The first effect is that a country's share of the total supply of goods changes and the second is that the value (the relative price) of the particular good a country produces changes. If $a_{\bar{i}}$ increases, the share of goods that country $i$ produces increases if $\bar{i} = i$ and decreases otherwise. The relative value of the good that country $i$ produces decreases if $j_{\bar{i}} = j_i$ and increases if $j_{\bar{i}} \neq j_i$.

In the first case above ($j_{\bar{i}} \neq j_i$), an increase in $a_{\bar{i}}$ decreases the total share of goods that a country produces but increases the relative price of the goods that the country produces. Which of the effects that dominates depends on the substitutability. For a low elasticity of substitution ($\sigma < 1$), the price effect dominates and the wealth share of the representative household in country $i$ increases. For the second case ($j_{\bar{i}} = j_i$ but $\bar{i} \neq i$), both effects go in the same direction and an increase in $a_{\bar{i}}$ decreases the wealth share of the representative household in country $i$. For the case $\bar{i} = i$, an increase in $a_i$ increases the total share of goods that the country produces but decreases the relative price of the goods that they produce, $j_i$. If country $i$ is the only country producing good $j_i$, then $A_{j_i} = l_i a_i$ and

$$\frac{ds_i}{da_i} = \frac{s_i l_i}{A_{j_i}} \frac{\sigma - 1}{\sigma} \left( 1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \right)$$

an expression that has the same sign as $\frac{\sigma-1}{\sigma}$. So this derivative has the opposite sign as compared to the case where $j_{\bar{i}} \neq j_i$. If country $i$ is not the only country producing good $j_i$, then $A_{j_i} > l_i a_i$ and the derivative is strictly positive for $\sigma = 1$ (and for any $\sigma \geq 1$). This reflects that the price effect of the productivity in country $i$ is weakened if other countries produce the same good and then more complementarity is needed for the price effect to dominate.

Putting together the size and share effects gives that

$$\frac{dc_i}{da_{\bar{i}}} = \begin{cases} \frac{1}{\sigma} \frac{s_i A l_{\bar{i}}}{A_{j_i}} \frac{A_{j_{\bar{i}}}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} & \text{if } j_{\bar{i}} \neq j_i \\ -\frac{1+\sigma}{\sigma} \frac{s_i A l_{\bar{i}}}{A_{j_i}} \left( 1 - \frac{A_{j_{\bar{i}}}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \right) & \text{if } j_{\bar{i}} = j_i \text{ but } \bar{i} \neq i \\ \frac{s_i A l_i}{A_{j_i}} \left( \frac{A_{j_i}}{l_i a_i} - \frac{1}{\sigma} \left( 1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \right) \right) & \text{if } \bar{i} = i \end{cases} \tag{4.12}$$

Given that welfare is strictly increasing in the consumption of the aggregate consumption good, the following proposition can be stated:

**Proposition 4.1.** *Consider how a change in $a_{\bar{i}}$ affects welfare in country $i$. If $\bar{i} \neq i$, then $\frac{du(c_i)}{da_{\bar{i}}}$ is negative if $j_i = j_{\bar{i}}$ and positive if $j_i \neq j_{\bar{i}}$. If $\bar{i} = i$ then*

$$\text{Sgn}\left(\frac{du(c_i)}{da_i}\right) = \text{Sgn}\left(\frac{A_{j_i}}{l_i a_i} - \frac{1}{\sigma}\left(1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}}\right)\right)$$

*Proof.* Since $\frac{A_{j_{\bar{i}}}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} < 1$ and $\frac{du(c_i)}{da_{\bar{i}}} = u'(c_i)\frac{dc_i}{da_{\bar{i}}}$, this immediately follows from (4.12). □

So the effect will be unambiguous if the productivity changes in other countries. The sign in the case where the countries' own productivity changes will depend on the parameters. In the expression for the derivative, $\frac{A_{j_i}}{l_i a_i} \geq 1$ while $1 - \frac{A_{j_i}^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{J} A_j^{\frac{\sigma-1}{\sigma}}} \in (0,1)$. This means that the derivative is positive if $\sigma \geq \frac{l_i a_i}{A_{j_i}} \leq 1$ (implying that the derivative is always positive if $\sigma \geq 1$). The ratio $\frac{l_i a_i}{A_{j_i}}$ gives the share of the total supply of good $j_i$ that is produced by country $i$. The smaller is this share, the weaker is the price effect of the productivity in country $i$ and the stronger the complementarity between the goods must be for the price effect to dominate. To get some further sense of when the effect is positive or negative, consider the case where $A_j = A$ for all $j$. Then, the outer parenthesis becomes

$$\frac{A}{l_i a_i} - \frac{1}{\sigma}\left(1 - \frac{1}{J}\right).$$

As before, this expression is always positive if $\frac{l_i a_i}{A} \leq \sigma$. If $\frac{l_i a_i}{A} > \sigma$, this is (weakly) negative if and only if $J \geq \frac{1}{1 - \sigma\frac{A}{l_i a_i}}$.

So the conclusion is that an increase in productivity in country $\bar{i}$ has a positive effect on welfare in country $i \neq \bar{i}$ if the countries are specialized in producing different goods, while it has a negative effect if the countries specialize in the production of the same type of goods. If the country's own productivity increases, the change in welfare due to the change in relative prices will tend to decrease the welfare of the representative household in the country. Counteracting this effect is the effect of an increase in the amount of goods produced. The net effect is ambiguous. So, in this setting it is perfectly possible that the derivative is negative. That would mean that the representative household would be better off in a situation where productivity decreases. The reason is that as the supplied quantity decreases, the relative price goes up. When the interpretation is that productivity decreases due to climate change, there would be likely to be other negative effects as well. There could be decreased productivity in the production of non-traded goods, or there could be other effects directly affecting welfare. So the possibility of a negative derivative should be interpreted with some caution. The effects on other countries should be much more robust.

### 4.2.4 The two-country case

Now, consider two countries. Country 1 can be considered as the developed countries or the north, whereas country 2 can be considered as the developing countries or the south. I will make the asymmetry between the countries stark by assuming that only country 2 is affected by climate change while all emissions of greenhouse gases are made by country 1. Under these assumptions, the question I will consider is how country 1 is indirectly affected by the effects in country 2. Furthermore, I will introduce trading costs in the form of import tariffs. The tariffs will then influence the welfare effects in country 1 of the climate induced changes in the productivity in country 2. Assuming that it is costly for country 1 to reduce the emissions of greenhouse gases, country 1 will then trade off the costs of emission reductions against the welfare effects of climate change. This trade off will depend on the tariffs and, therefore, the effect of changing tariffs on the optimal amount of greenhouse gas emissions in country 1 can be derived.

I will assume that there are two goods and that each country specializes in producing one of the goods. In terms of the model set up above, this means that $I = J = 2$, $\bar{I}_j = \{j\}$ for all $j$ and that $j_i = i$ for all $i$.

Assume now that climate change reduces productivity in country 2, $a_2$. From equation (4.12), we obtain

$$\frac{dc_1}{da_2} = \frac{1}{\sigma}\frac{s_1 A l_2}{A_1}\frac{A_2^{\frac{\sigma-1}{\sigma}}}{\sum_{j=1}^{2} A_j^{\frac{\sigma-1}{\sigma}}} > 0.$$

This implies that a decrease in productivity in country 2 decreases welfare in country 1. So, taking this channel into account should make country 1 want to reduce the emissions of greenhouse gases.

### 4.2.5 The two-country case with import tariffs

When trading goods, there are typically trading frictions. These can, among other things, be transportation costs, costs of adapting products to new markets, costs of marketing products on new markets and tariffs. The kind of trading costs that I will choose to model is import tariffs. There are two reasons for why I choose to consider tariffs. Tariffs are clearly determined by policy, meaning that they could be used strategically to affect the outcomes. In particular, higher import tariffs in the south could potentially be used as a threat to induce the north to reduce the emissions of greenhouse gases. Furthermore, tariffs do not lead to any production being destroyed. Therefore, the effects will only be driven by the distortionary effects of trading costs and not by the loss of resources.

### Model setup and equilibrium determination

Now assume that when households in country 1 buy good 2 a share $\tau_1$ is subtracted and that when households in country 2 buy good 1 a share $\tau_2$ is subtracted. Tariffs generate government revenues and I will assume that these are paid back lump sum to households. Let $T_i$ be the lump-sum transfer to households in country $i$.

I normalize the price of good 1: $p_1 = 1$.

For given trading costs, the definition of an equilibrium is the same as without trading costs. That is, the representative household in each country maximizes the utility from consumption and all production is consumed. I will not explicitly solve for prices and quantities here. Instead, I will derive a set of equations that implicitly determines the equilibrium quantities. This will allow me to derive comparative statics results.

The optimization problem for the representative household in country 1 is

$$\max_{c_{1,1},c_{1,2}} U(c_{1,1}, c_{1,2}) \text{ s.t. } a_1 + T_1 = c_{1,1} + \frac{p_2}{1-\tau_1} c_{1,2}.$$

The first-order conditions with respect to $c_{1,1}$ and $c_{1,2}$ give

$$\frac{p_2}{1-\tau_1} = \frac{U_2(c_{1,1}, c_{1,2})}{U_1(c_{1,1}, c_{1,2})} = \left(\frac{c_{1,2}}{c_{1,1}}\right)^{-\frac{1}{\sigma}} \Rightarrow p_2 = (1-\tau_1)\left(\frac{c_{1,2}}{c_{1,1}}\right)^{-\frac{1}{\sigma}}. \quad (4.13)$$

Similarly, the optimization problem for the representative household in country 2 is

$$\max_{c_{2,1},c_{2,2}} U(c_{2,1}, c_{2,2}) \text{ s.t. } p_2 a_2 + T_2 = \frac{c_{2,1}}{1-\tau_2} + p_2 c_{2,2}.$$

The first-order conditions with respect to $c_{2,1}$ and $c_{2,2}$ give

$$p_2(1-\tau_2) = \frac{U_2(c_{2,1}, c_{2,2})}{U_1(c_{2,1}, c_{2,2})} = \left(\frac{c_{2,2}}{c_{2,1}}\right)^{-\frac{1}{\sigma}} \Rightarrow p_2 = \frac{1}{1-\tau_2}\left(\frac{c_{2,2}}{c_{2,1}}\right)^{-\frac{1}{\sigma}}. \quad (4.14)$$

Assuming that all tariff revenues are paid back lump sum to the domestic households, the lump-sum transfers are

$$T_1 = \frac{\tau_1}{1-\tau_1} p_2 c_{1,2} \text{ and } T_2 = \frac{\tau_2}{1-\tau_2} c_{2,1}.$$

Substituting these in the budget constraints gives

$$\begin{aligned} a_1 &= c_{1,1} + p_2 c_{1,2} \\ p_2 a_2 &= c_{2,1} + p_2 c_{2,2}. \end{aligned}$$

Since the trading costs do not result in any loss of resources, it follows that the aggregate resource constraints are $l_i a_i = l_1 c_{1,i} + l_2 c_{2,i}$ for both $i$. Substituting for $p_2$ from (4.13) and (4.14) in the budget constraints of country 1 and 2, respectively, and combining with the aggregate resource constraints, gives the following characterization of the equilibrium:

$$a_1 = c_{1,1} + (1-\tau_1) c_{1,1}^{\frac{1}{\sigma}} c_{1,2}^{\frac{\sigma-1}{\sigma}} \tag{4.15}$$

$$a_2 = (1-\tau_2) c_{2,2}^{\frac{1}{\sigma}} c_{2,1}^{\frac{\sigma-1}{\sigma}} + c_{2,2} \tag{4.16}$$

$$a_1 = c_{1,1} + \frac{l_2}{l_1} c_{2,1} \tag{4.17}$$

$$a_2 = \frac{l_1}{l_2} c_{1,2} + c_{2,2}. \tag{4.18}$$

From these, it immediately follows that

$$c_{2,1} = \frac{l_1}{l_2}(1-\tau_1) c_{1,1}^{\frac{1}{\sigma}} c_{1,2}^{\frac{\sigma-1}{\sigma}} \tag{4.19}$$

$$c_{1,2} = \frac{l_2}{l_1}(1-\tau_2) c_{2,2}^{\frac{1}{\sigma}} c_{2,1}^{\frac{\sigma-1}{\sigma}}. \tag{4.20}$$

Therefore, equations (4.15)-(4.18) implicitly determine $c_{i,j}$ as functions of $a_1$, $a_2$, $\tau_1$ and $\tau_2$.

### Changing productivities and tariffs

I can now look at the effects of changes in $a_1$, $a_2$, $\tau_1$ and $\tau_2$. To see, for example, what is the effect of changing $a_1$, I treat $c_{i,j}$, given by the equilibrium conditions (4.15)-(4.18), as functions of $a_1$, differentiate the equilibrium conditions implicitly with respect to $a_1$ and solve for the derivatives of $c_{i,j}$ with respect to $a_1$. The derivatives with respect to $a_2$, $\tau_1$ and $\tau_2$ can be found in the same way. I perform these calculations in appendix 4.A.1.

Given all these derivatives, the changes in the consumption of the aggregate consumption good in country $i$, as $x \in \{a_1, a_2, \tau_1, \tau_2\}$ changes, can be calculated as

$$\frac{\partial c_i}{\partial x} = \left(\frac{c_i}{c_{i,1}}\right)^{\frac{1}{\sigma}} \frac{\partial c_{i,1}}{\partial x} + \left(\frac{c_i}{c_{i,2}}\right)^{\frac{1}{\sigma}} \frac{\partial c_{i,2}}{\partial x}. \tag{4.21}$$

I calculate this for each of the four parameters in appendix 4.A.1. The results for changes in $a_1$ are

$$\frac{\partial c_1}{\partial a_1} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}} \frac{\frac{a_2}{c_{2,2}} + \frac{\sigma-1}{\sigma}\left(1+\left(\frac{a_1}{c_{1,1}}-1\right)\frac{1}{1-\tau_1}\right)}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)}$$

$$\frac{\partial c_2}{\partial a_1} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}} \frac{l_1}{l_2}\frac{c_{1,2}}{c_{1,1}} \frac{\frac{1}{\sigma}\frac{1}{1-\tau_2}\frac{a_2}{c_{2,2}} + \frac{\tau_2}{1-\tau_2}\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)}.$$

The sign of the first derivative is ambiguous. If $\sigma \geq 1$, it is positive. If $\tau_1 = 0$, this can be shown to be equivalent to the condition in (4.12). The second derivative is unambiguously positive since $\frac{a_2}{c_{2,2}} \geq 1 \geq \tau_2$.

For changes in $a_2$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial a_2} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}} \frac{l_2}{l_1}\frac{c_{2,1}}{c_{2,2}} \frac{\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)}$$

$$\frac{\partial c_2}{\partial a_2} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}} \frac{\frac{a_1}{c_{1,1}} + \frac{\sigma-1}{\sigma}\left(1+\left(\frac{a_2}{c_{2,2}}-1\right)\frac{1}{1-\tau_2}\right)}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)}.$$

Here, the first derivative is unambiguously positive. This means that the result from section 4.2.4, that a decrease in the productivity in country 2 decreases welfare in country 1, holds also with trading costs. The sign of the second derivative is ambiguous in the same way as was the first derivative with respect to $c_1$.

For changes in $\tau_1$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial \tau_1} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}} \frac{l_2}{l_1} \frac{\frac{1}{\sigma}\frac{a_2}{c_{2,2}} + \frac{1-\sigma}{\sigma}\frac{\tau_1}{1-\tau_1}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)} \frac{c_{2,1}}{1-\tau_1}$$

$$\frac{\partial c_2}{\partial \tau_1} = -\left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}} \frac{l_1}{l_2} \frac{\frac{1}{1-\tau_2}\frac{1}{\sigma}\frac{a_2}{c_{2,2}} + \frac{\tau_2}{1-\tau_2}\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}-1\right)} \frac{c_{1,2}}{1-\tau_1}.$$

The sign of the first derivative is ambiguous. A sufficient condition for it to be positive is $\frac{1}{\sigma} > \frac{\sigma-1}{\sigma}\frac{\tau_1}{1-\tau_1} \Leftrightarrow \tau_1 < \frac{1}{\sigma}$. The second derivative is unambiguously negative since $\frac{a_2}{c_{2,2}} \geq 1 \geq \tau_2$.

For changes in $\tau_2$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial \tau_2} = -\left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}} \frac{l_2}{l_1} \frac{\frac{1}{1-\tau_1}\frac{1}{\sigma}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_2}$$

$$\frac{\partial c_2}{\partial \tau_2} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}} \frac{l_1}{l_2} \frac{\frac{1}{\sigma}\frac{a_1}{c_{1,1}} + \frac{1-\sigma}{\sigma}\frac{\tau_2}{1-\tau_2}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_2}$$

Here, the first derivative is unambiguously negative while the sign of the second derivative is ambiguous.

So if one of the countries increases its tariff, this always decreases welfare in the other country while the effect on the country's own welfare is ambiguous.

### Endogenizing climate

Consider now the situation where the changes in productivity are caused by the emissions of greenhouse gases made by country 1. An increase in emissions translates into a decrease in productivity in country 2. Emissions $E$ are valued by country 1 as $V(E)$. The total utility of country 1 is then[1]

$$V(E) + u(c_1),$$

where $V$ is assumed to be such that

$$V' > 0 \text{ and } V'' < 0. \tag{4.22}$$

For given productivities and tariffs, $c_1$ is determined competitively according to the conditions (4.15)-(4.18). Assume further that $a_1$ is exogenously given and that $a_2$ is a function of $E$.

I will make the following assumption about the productivity in country 2 as a function of $E$:

$$\frac{da_2}{dE} < 0 \text{ and } \frac{d^2a_2}{dE^2} \leq 0. \tag{4.23}$$

The equilibrium aggregate consumption in country 1, $c_1$, is now a function of $\tau_1$, $\tau_2$ and $E$. For given tariffs, the optimal choice of $E$ for country 1 is given by

$$\max_E V(E) + u(c_1).$$

[1] One motivation for this could be a two-period model where fossil fuels are only used in the first period and only by country 1. In the first period, productivity is given but in the second period, productivity depends on the first-period fossil-fuel use.

The first-order condition is

$$V'(E) + u'(c_1)\frac{\partial c_1}{\partial a_2}\frac{da_2}{dE} = 0. \tag{4.24}$$

This condition implicitly defines the amount of emissions made by country 1 for given tariffs.

Consider now how the choice of $E$ would change if one of the tariffs, $\tau_i$, were to change. Imposing the optimality condition (4.24), $E$ is now a function of $\tau_1$ and $\tau_2$. Differentiating the optimality condition (4.24) implicitly with respect to $\tau_i$ gives

$$\begin{aligned} 0 = {} & V''(E)\frac{\partial E}{\partial \tau_i} + u''(c_1)\left[\frac{\partial c_1}{\partial \tau_i} + \frac{\partial c_1}{\partial a_2}\frac{da_2}{dE}\frac{\partial E}{\partial \tau_i}\right]\frac{\partial c_1}{\partial a_2}\frac{da_2}{dE} \\ & + u'(c_1)\left[\frac{\partial^2 c_1}{\partial \tau_i \partial a_2} + \frac{\partial^2 c_1}{\partial a_2^2}\frac{da_2}{dE}\frac{\partial E}{\partial \tau_i}\right]\frac{da_2}{dE} + u'(c_1)\frac{\partial c_1}{\partial a_2}\frac{d^2 a_2}{dE^2}\frac{\partial E}{\partial \tau_i}. \end{aligned}$$

Solving for $\frac{\partial E}{\partial \tau_i}$ gives

$$\frac{\partial E}{\partial \tau_i} = -\frac{u''(c_1)\frac{\partial c_1}{\partial \tau_i}\frac{\partial c_1}{\partial a_2} + u'(c_1)\frac{\partial^2 c_1}{\partial \tau_i \partial a_2}}{V''(E) + \left[u''(c_1)\left(\frac{\partial c_1}{\partial a_2}\right)^2 + u'(c_1)\frac{\partial^2 2 c_1}{\partial a_2^2}\right]\left(\frac{da_2}{dE}\right)^2 + u'(c_1)\frac{\partial c_1}{\partial a_2}\frac{d^2 a_2}{dE^2}}\frac{da_2}{dE}.$$

To simplify the analysis, I will assume that the tariffs are small so that I can evaluate all derivatives for $\tau_1 = \tau_2 = 0$. Given assumptions (4.23) and (4.24), and using that $\frac{\partial c_1}{\partial a_2} > 0$, the first and third terms are both negative. The second term (see appendix 4.A.2 for calculations) is

$$u''(c_1)\left(\frac{\partial c_1}{\partial a_2}\right)^2 + u'(c_1)\frac{\partial^2 c_1}{\partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial a_2}\frac{1}{c_{2,2}}\frac{\frac{1-\theta-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2 - \frac{1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}\right)^2}.$$

A sufficient condition for this to be negative is that that $\theta \geq 1$. I will assume that the denominator is negative. Combined with $\frac{da_2}{dE} < 0$, as assumed in (4.23), this implies that

$$\text{Sgn}\left(\frac{\partial E}{\partial \tau_i}\right) = \text{Sgn}\left(-u''(c_1)\frac{\partial c_1}{\partial \tau_i}\frac{\partial c_1}{\partial a_2} - u'(c_1)\frac{\partial^2 c_1}{\partial \tau_i \partial a_2}\right).$$

For $\tau_1$ (see appendix 4.A.2), we obtain

$$u''(c_1)\frac{\partial c_1}{\partial \tau_1}\frac{\partial c_1}{\partial a_2} + u'(c_1)\frac{\partial^2 c_1}{\partial \tau_1 \partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial \tau_1}\frac{1}{c_{2,2}}\frac{\frac{2-\theta-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2 + \frac{\sigma-1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}\right)^2}.$$

When $\tau_1 = 0$, $\frac{\partial c_1}{\partial \tau_1} > 0$ and

$$\mathrm{Sgn}\left(\frac{\partial E}{\partial \tau_1}\right) = \mathrm{Sgn}\left(\frac{\theta + \sigma - 2}{\sigma}\frac{a_1}{c_{1,1}} + \frac{1-\sigma}{\sigma}\frac{a_2}{c_{2,2}}\right).$$

For $\tau_2$ (see appendix 4.A.2), we arrive at

$$u''(c_1)\frac{\partial c_1}{\partial \tau_2}\frac{\partial c_1}{\partial a_2} + u'(c_1)\frac{\partial^2 c_1}{\partial \tau_2 \partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial \tau_2}\frac{1}{c_{2,2}}\frac{\frac{1-\theta}{\sigma}\left(\frac{a_1}{c_{1,1}} - 1\right) + 2\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}.$$

When $\tau_1 = 0$, $\frac{\partial c_1}{\partial \tau_2} < 0$ and

$$\mathrm{Sgn}\left(\frac{\partial E}{\partial \tau_2}\right) = \mathrm{Sgn}\left(\frac{1-\theta}{\sigma}\left(\frac{a_1}{c_{1,1}} - 1\right) + 2\frac{\sigma-1}{\sigma}\right).$$

These results can be summarized in the following proposition:

**Proposition 4.2.** *Assume that $\theta \geq 1$ and that the tariffs are small ($\tau_1 \approx \tau_2 \approx 0$). Then, if the emissions of greenhouse gases are given by* (4.24)*, they depend on the tariffs as*

$$\mathrm{Sgn}\left(\frac{\partial E}{\partial \tau_1}\right) = \mathrm{Sgn}\left(\frac{\theta + \sigma - 2}{\sigma}\frac{a_1}{c_{1,1}} + \frac{1-\sigma}{\sigma}\frac{a_2}{c_{2,2}}\right)$$
$$\mathrm{Sgn}\left(\frac{\partial E}{\partial \tau_2}\right) = \mathrm{Sgn}\left(\frac{1-\theta}{\sigma}\left(\frac{a_1}{c_{1,1}} - 1\right) + 2\frac{\sigma-1}{\sigma}\right)$$

*Proof.* Follows from the above calculations. □

The result for changes in $\tau_2$ implies that, if e.g. $\theta \geq 1$ and $\sigma \leq 1$, country 2 could induce country 1 to reduce its emissions by threatening with tariffs in the future. Since country 2 would gain from imposing tariffs, the threat would also be credible. Naturally, there are many other strategic considerations that go into the setting of tariffs. Note that, without further considerations, the credibility problem is rather that it would not be credible for country 2 to promise not to impose tariffs if country 1 reduces its emissions.

### 4.2.6 Climate change mitigation policy

The above suggested interpretation of the changes in productivity was that they were caused by climate change. An alternative interpretation is that productivity changes due to regulations to reduce greenhouse gas emissions. That is, productivity decreases due to climate change mitigation policy. There should be similar effects from that kind of changes in productivity. So, the general equilibrium effects from trade should also be included in calculations of the costs of mitigation policy. Proposition 4.1 implies that mitigation policy that reduces productivity in the north will decrease welfare in the south. As for the climate related effects, the proposition does not rule out that a mitigation policy could increase welfare in the north. While there could still be adverse effects from the non-traded sector, the other adverse effects of climate change would not be present here.

This is the situation with which the carbon leakage literature is concerned and carbon leakage could provide counteracting benefits for unregulated countries. Carbon leakage can occur in different ways. One way is through the fossil-fuel price. If regulation in some countries decreases the world market fossil-fuel price, this should increase the fossil-fuel use in other countries and it should benefit those countries. A different way in which carbon leakage can materialize is if production is relocated to countries with a less stringent regulation. To the extent that this occurs, this should benefit those countries to which production moves. In terms of the many countries model, countries to which production can move could be interpreted as producing the same good as the countries that production moves from and proposition 4.1 then gives that the indirect effects are positive for these countries. The empirical pollution haven literature (see Copeland and Taylor, 2004, for an overview) typically finds that other factors are more important than regulation for production location decisions. This seems to suggest that there should be limited benefits from production relocation in unregulated countries.

## 4.3 Insurance against weather variability

I will now, instead, consider the effects of changes in the distribution of weather related shocks. Climate change is predicted to not just increase average temperatures but also to change the distribution of weather outcomes. In particular, extreme weather events are predicted to become more frequent and more severe (IPCC 2007). Since these shocks will not be perfectly correlated across the world, there will be scope for insuring against such shocks through trade in financial instruments. Since the distribution of weather outcomes will depend on the amount of climate change, so will the world market prices of the financial instruments that can be used to insure against these shocks. Therefore, as in the case of trade in goods, countries will be indirectly affected by the changes in the weather distribution, and these indirect effects will go through changes in world market prices.

I will analyze these effects in the context of a many-country, two-period, endowment model. The first-period endowments are deterministic while the second-period endowments are stochastic. In the first period, there is trade in a complete set of financial instruments with state contingent second-period payoffs.

In section 4.3.1 I will set up the model. In section 4.3.2 I will solve for the equilibrium allocation. I will then, in section 4.3.3, consider how changes in the endowment distribution affect welfare. In section 4.3.4, I will consider what the general results imply for the two-country (north and south) case.

### 4.3.1 Model setup

Consider a two-period model. There are $I$ countries and country $i$ has population $l_i$. In the first period, the representative household in each country $i$ receives a deterministic endowment $y_{1,i}$. In the second period, there are $N$ possible states. In state $n$, the representative household receives an endowment $y_{2,i,n}$. The probability of the second-period state being $n$ is $\pi_n$, where $\sum_{n=1}^{N} \pi_n = 1$. Changes in the distribution of weather-induced shocks will be modeled as changes in $\{\pi_n\}$ and $\{y_{2,i,n}\}$.

Let

$$Y_1 = \sum_{i=1}^{I} l_i y_{1,i} \text{ and } Y_{2,n} = \sum_{i=1}^{I} l_i y_{2,i,n}$$

and

$$s_{i,n} = \frac{y_{2,i,n}}{Y_{2,n}}. \tag{4.25}$$

$Y_1$ is the total endowment in period 1. $Y_{2,n}$ is the total endowment in the second-period state $n$ and $s_{i,n}$ is the share of that endowment that goes to the representative household in country $i$, that is, the share of total endowment in state $n$ that the representative household in country $i$ receives.

Let the consumption of the representative household in country $i$ be $c_{1,i}$ in period 1 and $c_{2,i,n}$ in period 2 if the state is $n$. Households value an allocation as the discounted expected utility from consumption:

$$V_i = u(c_{1,i}) + \beta E_1 \left[ u(c_{2,i,n}) \right] = u(c_{1,i}) + \beta \sum_{n=1}^{N} \pi_n u(c_{2,i,n}), \tag{4.26}$$

where

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}. \tag{4.27}$$

There are complete markets for insurance against the second-period uncertainty. For each state $n$, there is an asset that pays one unit of consumption in the second period if the state is $n$ and 0 in all other states. Let $b_{i,n}$ be the holdings of the representative household in country

$i$ of the asset that pays in state $n$ and let $q_n$ be the price, in period 1, of the asset that pays in state $n$.

The asset holdings must fulfill the market clearing condition:

$$\sum_{i=1}^{I} l_i b_{i,n} = 0 \text{ for all } n. \tag{4.28}$$

Consumption is given by

$$c_{1,i} = y_{1,i} - \sum_{n=1}^{N} q_n b_{i,n} \text{ and } c_{2,i,n} = y_{2,i,n} + b_{i,n}. \tag{4.29}$$

### 4.3.2 Equilibrium

I will now solve for the equilibrium allocation. Households are assumed to act as price takers and buy assets to maximize discounted expected utility. As in the case of trade in goods, the price-taking assumption is important since the indirect effects go through changes in prices.

An equilibrium is a set of prices $\{q_n\}_{n=1}^{N}$ and asset holdings $\{b_{i,n}\}_{i,n}$ such that

- households maximize utility (4.26) with consumption given by (4.29)
- the asset holdings fulfill the market clearing condition (4.28) for all $n$

The optimization problem of the representative household in country $i$ is

$$\max_{\{b_{i,n}\}_{n=1}^{N}} u\left(y_{1,i} - \sum_{n=1}^{N} q_n b_{i,n}\right) + \beta \sum_{n=1}^{N} \pi_n u\left(y_{2,i,n} + b_{i,n}\right).$$

In principle, there should be non-negativity constraints on consumption. However, since $\lim_{c\to 0} u'(c) = \infty$, the non-negativity constraints will never bind.

The first-order condition with respect to $b_{i,n}$ is

$$q_n u'(c_{1,i}) = \beta\pi_n u'(c_{2,i,n}) \Rightarrow q_n = \beta\pi_n \frac{u'(c_{2,i,n})}{u'(c_{1,i})}. \tag{4.30}$$

In this expression, only the ratio of marginal utilities depends on $i$. Thus, this ratio will be the same for all countries. The representative household balances the price of the assets against the marginal value of consumption in period 1 and in different states of period 2, taking the time preference and the realization probabilities into account.

Using the functional form of the utility function (4.27), we obtain

$$q_n = \beta\pi_n \frac{u'(c_{2,i,n})}{u'(c_{1,i})} = \beta\pi_n \left(\frac{c_{2,i,n}}{c_{1,i}}\right)^{-\theta} \Rightarrow c_{2,i,n} = \left(\frac{\beta\pi_n}{q_n}\right)^{\frac{1}{\theta}} c_{1,i}.$$

Substituting the expressions for consumption from (4.29), multiplying by $l_i$ and summing over $i$ on both sides gives

$$\begin{aligned}
\sum_{i=1}^{I} l_i \left(y_{2,i,n} + b_{i,n}\right) &= \sum_{i=1}^{I} \left(\frac{\beta\pi_n}{q_n}\right)^{\frac{1}{\theta}} l_i \left(y_{1,i} - \sum_{n'=1}^{N} q_{n'} b_{i,n'}\right) \\
\sum_{i=1}^{I} l_i y_{2,i,n} + \sum_{i=1}^{I} l_i b_{i,n} &= \left(\frac{\beta\pi_n}{q_n}\right)^{\frac{1}{\theta}} \sum_{i=1}^{I} l_i y_{1,i} - \left(\frac{\beta\pi_n}{q_n}\right)^{\frac{1}{\theta}} \sum_{n'=1}^{N} q_{n'} \sum_{i=1}^{I} l_i b_{i,n'} \\
\sum_{i=1}^{I} l_i y_{2,i,n} &= \left(\frac{\beta\pi_n}{q_n}\right)^{\frac{1}{\theta}} \sum_{i=1}^{I} l_i y_{1,i},
\end{aligned}$$

where the market clearing condition for asset holdings (4.28) was used for the last step.

Solving for $q_n$ gives

$$q_n = \beta\pi_n \left(\frac{\sum_{i=1}^{I} l_i y_{1,i}}{\sum_{i=1}^{I} l_i y_{2,i,n}}\right)^{\theta} = \beta\pi_n \left(\frac{Y_1}{Y_{2,n}}\right)^{\theta}. \tag{4.31}$$

We see that the price will be higher for assets that pay in states that are realized with high probability and states where the total endowment is small.

Comparing this to the first-order condition with respect to $b_{i,n}$ in (4.30) gives that, for all $i$ and $n$,

$$\frac{c_{1,i}}{c_{2,i,n}} = \frac{Y_1}{Y_{2,n}} \Rightarrow c_{2,i,n} = \frac{c_{1,i}}{Y_1} Y_{2,n}.$$

Now define

$$s_i = \frac{c_{1,i}}{Y_1};$$

this is the share of the total endowment that the representative household in country $i$ consumes in the first period and in each state in the second period.

From the consumption expressions (4.29), we see that

$$s_i Y_1 = c_{1,i} = y_{1,i} - \sum_{n=1}^{N} q_n b_{i,n} \Rightarrow y_{1,i} - s_i Y_1 = \sum_{n=1}^{N} q_n b_{i,n}$$

and that

$$s_iY_{2,n} = c_{2,i,n} = y_{2,i,n} + b_{i,n} \Rightarrow b_{i,n} = s_iY_{2,n} - y_{2,i,n}.$$

Combining these expressions gives

$$\begin{aligned} y_{1,i} - s_iY_1 &= \textstyle\sum_{n=1}^{N} q_n \left(s_iY_{2,n} - y_{2,i,n}\right) \\ \Rightarrow y_{i,1} + \textstyle\sum_{n=1}^{N} q_n y_{2,i,n} &= s_i \left(Y_1 + \textstyle\sum_{n=1}^{N} q_n Y_{2,n}\right). \end{aligned}$$

Thus, we arrive at

$$s_i = \frac{y_{i,1} + \sum_{n=1}^{N} q_n y_{2,i,n}}{Y_1 + \sum_{n=1}^{N} q_n Y_{2,n}}.$$

We see that $s_i$ is also the wealth share of the representative household in country $i$. The share can be rewritten, in terms of exogenous objects, as

$$\begin{aligned} s_i &= \frac{y_{i,1} + \sum_{n=1}^{N} \beta\pi_n \frac{Y_1^{\theta}}{Y_{2,n}^{\theta}} y_{2,i,n}}{Y_1 + \sum_{n=1}^{N} \beta\pi_n \frac{Y_1^{\theta}}{Y_{2,n}^{\theta}} Y_{2,n}} = \frac{y_{i,1} + Y_1^{\theta} \sum_{n=1}^{N} \beta\pi_n Y_{2,n}^{-\theta} y_{2,i,n}}{Y_1 + Y_1^{\theta} \sum_{n=1}^{N} \beta\pi_n Y_{2,n}^{-\theta} Y_{2,n}} \\ &= \frac{\frac{y_{i,1}}{Y_1} Y_1^{1-\theta} + \beta \sum_{n=1}^{N} \pi_n \frac{y_{2,i,n}}{Y_{2,n}} Y_{2,n}^{1-\theta}}{Y_1^{1-\theta} + \beta \sum_{n=1}^{N} \pi_n Y_{2,n}^{1-\theta}}. \end{aligned}$$

Using the second-period endowment share defined in (4.25), the wealth share becomes

$$s_i = \frac{\frac{y_{i,1}}{Y_1} Y_1^{1-\theta} + \beta \sum_{n=1}^{N} \pi_n s_{i,n} Y_{2,n}^{1-\theta}}{Y_1^{1-\theta} + \beta \sum_{n=1}^{N} \pi_n Y_{2,n}^{1-\theta}}. \tag{4.32}$$

For given $Y_1$ and $Y_{2,n}$, the wealth share is always increasing in $y_{1,i}$ and $y_{2,i,n}$. That is, a higher endowment for the representative household is always preferable given total endowments.

Assume that $Y_1^{1-\theta} + \beta \sum_{n=1}^{N} \pi_n Y_{2,n}^{1-\theta}$ is given. The wealth share, $s_i$, is increasing in each endowment share, $s_{i,n}$, but, comparing states, which states the country would like to have high shares in depends on the value of $\theta$. If $\theta > 1$, the country prefers to have a large share of the endowment, $s_{i,n}$, in states $n$ where $Y_{2,n}$ is small. If $\theta < 1$ the opposite holds. Having a large share in a state with a high endowment gives a higher endowment, but the value of that endowment is smaller since consumption in that state is valued less. Which of these effects that dominates depends on $\theta$.

Using the shares, consumption can be written

$$c_{1,i} = s_iY_1 \text{ and } c_{2,i,n} = s_iY_{2,n}.$$

The equilibrium asset holdings are

$$c_{2,i,n} = y_{2,i,n} + b_{i,n} \Rightarrow s_i Y_{2,n} = s_{i,n} Y_{2,n} + b_{i,n}.$$

Solving for the asset holdings gives

$$b_{i,n} = (s_i - s_{i,n}) Y_{2,n}. \tag{4.33}$$

Thus, the representative household holds positive amounts of the assets that pay in states where the endowment share is smaller than the wealth share, and the other way around. Note that it is possible for a country to hold negative (or positive) amounts of all assets if much (little) of the wealth comes from the period 1 endowment $y_{1,i}$.

In conclusion, the equilibrium is determined by the asset prices (4.31) and the asset holdings (4.33) with endowment shares and wealth shares given by (4.25) and (4.32), respectively. Using these, the allocation of consumption can be calculated from (4.29).

### 4.3.3 Welfare effects of changing the second-period distribution

Starting from the equilibrium allocation for a given distribution, that is given values of $\pi_n$, $y_{1,i}$ and $y_{2,i,n}$, we can now look at how changes in the distribution affect welfare. A change in the probabilities, $\pi_n$, can be considered as a change in the probability of an event such as a heatwave or a storm occurring. A change in the endowment in a given state, $y_{2,i,n}$, can be considered as a change in the severity of such an event. To be able to distinguish more easily between the direct effects and the effects through changing equilibrium prices, we can start from an expression with consumption from (4.29) substituted in the expression for welfare (4.26), namely,

$$V_i = u\left(y_{1,i} - \sum_{n=1}^{N} q_n b_{i,n}\right) + \beta \sum_{n=1}^{N} \pi_n u(y_{2,i,n} + b_{i,n}),$$

where prices $q_n$ and asset holdings $b_{i,n}$ are at their equilibrium values. Consider now a change $\Delta$ that affects probabilities and second-period endowments. Since asset prices and asset holdings depend on the endowment distribution, these will change endogenously in response to the change $\Delta$. The change in welfare is given by

$$\begin{aligned}\frac{dV_i}{d\Delta} &= -u'(c_{1,i})\sum_{i=1}^{N}\left(\frac{dq_n}{d\Delta}b_{i,n} + q_n\frac{db_{i,n}}{d\Delta}\right) + \beta\sum_{n=1}^{N}\frac{d\pi_n}{d\Delta}u(c_{2,i,n}) \\ &\quad + \beta\sum_{n=1}^{N}\pi_n u'(c_{2,i,n})\left(\frac{dy_{2,i,n}}{d\Delta} + \frac{db_{i,n}}{d\Delta}\right) \\ &= -u'(c_{1,i})\sum_{i=1}^{N}\frac{dq_n}{d\Delta}b_{i,n} + \beta\sum_{n=1}^{N}\frac{d\pi_n}{d\Delta}u(c_{2,i,n}) + \beta\sum_{n=1}^{N}\pi_n u'(c_{2,i,n})\frac{dy_{2,i,n}}{d\Delta} \\ &\quad + \sum_{n=1}^{N}\left(\beta\pi_n u'(c_{2,i,n}) - u'(c_{1,i})q_n\right)\frac{db_{i,n}}{d\Delta}.\end{aligned}$$

Using the first-order condition with respect to $b_{i,n}$ (4.30), the change in welfare is

$$\frac{dV_i}{d\Delta} = \beta\sum_{n=1}^{N}\frac{d\pi_n}{d\Delta}u(c_{2,i,n}) + \beta\sum_{n=1}^{N}\pi_n u'(c_{2,i,n})\frac{dy_{2,i,n}}{d\Delta} - u'(c_{1,i})\sum_{n=1}^{N}\frac{dq_n}{d\Delta}b_{i,n}. \tag{4.34}$$

The effect of the change $\Delta$ on welfare can be divided into two direct effects, the first two effects in the last expression, and an indirect price effect.

The direct effect of the changes in probabilities is such that a redistribution of the probability mass from states with low consumption to states with high consumption increases welfare. Note that the changes in probabilities must sum to 0. For all countries, the states with high consumption are the states with a high combined endowment, $Y_{2,n}$.

The direct effect of changes in endowments is such that an increase in the endowment in any state increases welfare.

The price effect is such that welfare increases from price increases (decreases) of assets of which the household holds negative (positive) amounts. From (4.33), the household holds positive (negative) amounts of the assets that pay in states where the household's endowment share is smaller (larger) than the wealth share. From the expression for the prices (4.31), it follows that prices of the asset that pays in state $n$ increase in $\pi_n$ and decrease in $Y_{2,n}$.

To see what is the total effect, I will consider changes in probabilities and endowments separately.

## Changes in realization probabilities

Start by assuming that $\frac{dy_{2,i,n}}{d\Delta} = 0$ for all $i$ and $n$. Using the expression for prices (4.31), the welfare effect given by (4.34) becomes

$$\begin{aligned}
\frac{dV_i}{d\Delta} &= \beta \sum_{n=1}^{N} \frac{d\pi_n}{d\Delta} u(c_{2,i,n}) - u'(c_{1,i}) \sum_{n=1}^{N} \frac{dq_n}{d\Delta} b_{i,n} \\
&= \beta \sum_{n=1}^{N} \frac{d\pi_n}{d\Delta} u(c_{2,i,n}) - u'(c_{1,i}) \sum_{n=1}^{N} \frac{q_n}{\pi_n} \frac{d\pi_n}{d\Delta} b_{i,n} \\
&= \sum_{n=1}^{N} \left[ \beta u(c_{2,i,n}) - u'(c_{1,i}) \frac{q_n}{\pi_n} b_{i,n} \right] \frac{d\pi_n}{d\Delta} \\
&= \sum_{n=1}^{N} \left[ \beta u(c_{2,i,n}) - \beta u'(c_{2,i,n}) b_{i,n} \right] \frac{d\pi_n}{d\Delta} \\
&= \beta \sum_{n=1}^{N} \left[ u\left(s_i Y_{2,n}\right) - u'\left(s_i Y_{2,n}\right) (s_i - s_{i,n}) Y_{2,n} \right] \frac{d\pi_n}{d\Delta}.
\end{aligned}$$

For logarithmic utility, $\theta = 1$, this expression becomes

$$\begin{aligned}
\frac{dV_i}{d\Delta} &= \beta \sum_{n=1}^{N} \left[ \log\left(s_i Y_{2,n}\right) - \frac{(s_i - s_{i,n}) Y_{2,n}}{s_i Y_{2,n}} \right] \frac{d\pi_n}{d\Delta} \\
&= \beta \sum_{n=1}^{N} \left[ \log\left(Y_{2,n}\right) - \frac{(s_i - s_{i,n})}{s_i} \right] \frac{d\pi_n}{d\Delta} + \beta \log(s_i) \sum_{n=1}^{N} \frac{d\pi_n}{d\Delta} \\
&= \beta \sum_{n=1}^{N} \left[ \log\left(Y_{2,n}\right) + \frac{s_{i,n}}{s_i} - 1 \right] \frac{d\pi_n}{d\Delta} \\
&= \beta \sum_{n=1}^{N} \left[ \log\left(Y_{2,n}\right) + \frac{s_{i,n}}{s_i} \right] \frac{d\pi_n}{d\Delta}. \qquad (4.35)
\end{aligned}$$

We see that the representative household would prefer a redistribution of the probability mass to states with a higher total endowment $Y_{2,n}$ and a higher endowment share $s_{i,n}$.

For $\theta \neq 1$, the expression becomes

$$\begin{aligned}\frac{dV_i}{d\Delta} &= \beta \sum_{n=1}^{N} \left[ \frac{c_{2,i,n}^{1-\theta} - 1}{1-\theta} - c_{2,i,n}^{-\theta} b_{i,n} \right] \frac{d\pi_n}{d\Delta} \\ &= \beta \sum_{n=1}^{N} \left[ \frac{\left(s_i Y_{2,n}\right)^{1-\theta}}{1-\theta} - \left(s_i Y_{2,n}\right)^{-\theta} (s_i - s_{i,n}) Y_{2,n} \right] \frac{d\pi_n}{d\Delta} \qquad (4.36) \\ &\quad - \frac{\beta}{1-\theta} \sum_{n=1}^{N} \frac{d\pi_n}{d\Delta} \\ &= \beta \sum_{n=1}^{N} \left(s_i Y_{2,n}\right)^{1-\theta} \left[ \frac{1}{1-\theta} - \frac{s_i - s_{i,n}}{s_i} \right] \frac{d\pi_n}{d\Delta} \\ &= \beta s_i^{1-\theta} \sum_{n=1}^{N} Y_{2,n}^{1-\theta} \left[ \frac{1}{1-\theta} - \frac{s_i - s_{i,n}}{s_i} \right] \frac{d\pi_n}{d\Delta} \\ &= \beta s_i^{1-\theta} \sum_{n=1}^{N} Y_{2,n}^{1-\theta} \left[ \frac{\theta}{1-\theta} + \frac{s_{i,n}}{s_i} \right] \frac{d\pi_n}{d\Delta}. \qquad (4.37)\end{aligned}$$

Here, the welfare effects are more complicated. Comparing two states with the same total endowment $Y_{2,n}$, a state with a higher endowment share $s_{i,n}$ is better. If $\theta < 1$, states with a larger total endowment $Y_{2,n}$ are better. If $\theta > 1$, a larger $Y_{2,n}$ is better (worse) if $s_{i,n}$ is small (large) enough so that the parenthesis is negative (positive). If the endowment share is large, the household can benefit from a small total endowment since that would increase the price of assets that the household sells.

This can all be summarized in the following proposition:

**Proposition 4.3.** *For logarithmic utility, $\theta = 1$, the welfare of the representative household in country $i$ is increased if the probability mass is redistributed towards second-period states $n$ such that $\left[\log\left(Y_{2,n}\right) + \frac{s_{i,n}}{s_i}\right]$ is large. For $\theta \neq 1$, the welfare of the representative household in country $i$ is increased if the probability mass is redistributed towards second-period states $n$ such that $Y_{2,n}^{1-\theta}\left[\frac{\theta}{1-\theta} + \frac{s_{i,n}}{s_i}\right]$ is large.*

*Proof.* Follows from (4.35) and (4.37). □

This concludes the description of the welfare effects of varying the realization probabilities $\{\pi_n\}_{n=1}^{N}$.

**Changes in second-period endowments**

Consider now instead the case where $\frac{d\pi_n}{d\Delta} = 0$ for all $n$. The effects of changes in endowments can be analyzed for an individual endowment. So I will consider a change in the endowment $y_{2,i',n}$.

From (4.31), the price change induced by a change in $y_{2,i',n}$ is

$$\frac{dq_n}{dy_{2,i',n}} = -\theta \frac{q_n}{Y_{2,n}} \frac{dY_{2,n}}{dy_{2,i',n}} = -\theta \frac{q_n}{Y_{2,n}} l_{i'}.$$

Using this in (4.34), the welfare effect, for the representative household in country $i$, from a change in $y_{2,i',n}$ is

$$\begin{aligned}\frac{dV_i}{dy_{2,i',n}} &= \beta\pi_n u'(c_{2,i,n}) \frac{dy_{2,i,n}}{dy_{2,i',n}} - u'(c_{1,i}) \frac{dq_n}{dy_{2,i',n}} b_{i,n} \\ &= \beta\pi_n u'(c_{2,i,n}) \frac{dy_{2,i,n}}{dy_{2,i',n}} + u'(c_{1,i})\theta \frac{q_n}{Y_{2,n}} l_{i'} b_{i,n}.\end{aligned}$$

The first, direct, effect is positive if $i' = i$ and zero otherwise. The second, indirect, effect has the same sign as $b_{i,n}$. If $y_{2,i',n}$ increases, this decreases the price $q_n$ and this is positive or negative depending on the sign of $b_{i,n}$. So, if $i' \neq i$, the total effect only depends on $b_{i,n}$. If $i' = i$, we see that

$$\begin{aligned}\frac{dV_i}{dy_{2,i,n}} &= \beta\pi_n u'(c_{2,i,n}) + \theta u'(c_{1,i}) \frac{q_n}{Y_{2,n}} l_i b_{i,n} = \{\text{FOC}\} = \\ &= \beta\pi_n u'(c_{2,i,n}) + \theta\beta\pi_n u'(c_{2,i,n}) \frac{b_{i,n}}{Y_{2,n}} l_i \\ &= \beta\pi_n u'(c_{2,i,n}) \left(1 + \theta \frac{(s_i - s_{i,n})Y_{2,n}}{Y_{2,n}} l_i\right) \\ &= \beta\pi_n u'(c_{2,i,n}) \left(1 + \theta(s_i - s_{i,n}) l_i\right).\end{aligned}$$

The derivative can be negative if $\theta$ is large and $s_{i,n}$ is larger than $s_i$. However, quantitatively, this does not seem very likely since it would require a large difference between $s_i$ and $s_{i,n}$. If, e.g., $\theta = 2$ it would require $(s_i - s_{i,n})l_i > 0.5$.

This discussion is summarized in the following proposition:

**Proposition 4.4.** *The welfare effects for the representative household in country $i$ from a change in $y_{2,i',n}$ satisfy*

$$\text{Sgn}\left(\frac{dV_i}{dy_{2,i',n}}\right) = \begin{cases} \text{Sgn}(b_{i,n}) & \text{if } i' \neq i \\ \text{Sgn}\left(1 + \theta(s_i - s_{i,n}) l_i\right) & \text{if } i' = i \end{cases}$$

*with $b_{i,n}$ given by* (4.33)*, $s_i$ given by* (4.32) *and $s_{i,n}$ given by* (4.25)*.*

*Proof.* Follows from the calculations above. □

This concludes the description of the welfare effects of changes in $\{y_{2,i,n}\}$.

### 4.3.4 The two-country case

Consider now the two-country case similar to that discussed in section 4.2.4. It is not possible to have changes in the realization probabilities that only affect country 2. So I will only consider changes in the second-period endowments in country 2. The welfare effects in country 1 are

$$\frac{dV_1}{d\Delta} = -u'(c_{1,1}) \sum_{n=1}^{N} \frac{dq_n}{d\Delta} b_{1,n} = u'(c_{1,1}) \sum_{n=1}^{N} \theta \frac{q_n}{Y_{2,n}} l_2 b_{1,n} \frac{dy_{2,2,n}}{d\Delta}$$
$$= u'(c_{1,1})\theta l_2 \sum_{n=1}^{N} \frac{q_n}{Y_{2,n}} b_{1,n} \frac{dy_{2,2,n}}{d\Delta}$$

The sign of the effect will depend on the pattern of changes in endowments. One prediction is that climate change will increase the severity of extreme events. If this is interpreted as that endowments will decrease the most in states where the endowments $y_{2,2,n}$ are small to begin with, it seems likely that these will be states where $b_{1,n} < 0$ since country 2 would want to buy an insurance against these outcomes. This would make the welfare effect on country 1 positive.

So, the insurance instruments channel would tend to make country 1 less interested in reducing the emissions of greenhouse gases.

## 4.4 Conclusions and discussion

This chapter has highlighted two ways in which countries are economically linked to each other and how these links affect the calculations of gains and losses associated with climate change. These channels imply that a country that is not directly affected by climate change will still indirectly be affected through changes in world market prices. The general conclusion is that if a country is a net seller of a good (or a financial instrument), changes that increase the demand or decrease the supply of this good will benefit the country since this will increase the relative value of the goods that the country sells. For goods of which the country is a net buyer, the opposite will hold. In the stylized two-country example, this implies that looking at the channel through trade in goods, the north would be hurt by decreased productivity in the south. When looking at trade in financial instruments that can be used to insure against weather variability, the north will tend to gain if the severity of the extreme bad weather events in the south increases.

In both these cases, all agents have been assumed to be price takers. This assumption is important since the indirect effects go through changing world market prices. When looking at trade in goods, a country that is negatively affected by climate change will also experience an offsetting effect since the relative price of the goods that it produces will increase. This offsetting effect captures the potential to increase income through mark-up pricing. So, if prices are not competitively set to begin with, these indirect effects will be different. This is something that would be interesting to investigate further.

In the parts on insurance, I have assumed complete markets. This is not a particularly realistic assumption. Arrow et al. (1996) point out that insurance companies may be reluctant to sell insurance against events with unknown probability distributions. While this may very well be true, the instruments discussed here need not literally be insurance. A different version of the model with endogenous investments in capital could be set up. In that model, the second-period distribution could be on productivity of capital in the countries. Such a model gives similar results and allowing for direct investments in capital in other countries gives complete markets. Chichilnisky (1998) argues that if there is uncertainty about the future distribution of events, this will mean that instruments contingent on these events will not provide complete markets. There are also effects of climate change that are not tied to productivities of capital. So, it might be reasonable to assume that markets are not complete. My guess is, however, that this would only be problematic for the results derived here if there is a systematic relationship between market imperfections and the effects of climate change.

For tractability and to highlight the basic mechanisms, some important aspects have been omitted from the analysis. In the model with trade in goods, the comparative advantages were assumed to be exogenously given and regardless of climate change, the comparative advantages are strong enough to induce the countries to specialize in one given good. To the extent that the comparative advantages are still such that the signs of net exports of the goods are the same, regardless of climate change, this does not seem very problematic. The indirect effects going through prices will still be such that price increases (decreases) benefit countries that are net exporters (importers) of the good. One case where this might not be the case is agricultural productivity. The growing zones for various crops will tend to move away from the equator. So for some goods, the comparative advantages may change directions. This suggests that having good-specific productivities, endogenous choices of what to produce and considering changes in the good-specific productivities could add an interesting structure to the indirect effects.

Another aspect that has been completely omitted from the analysis is that the effects of climate change are endogenous. In the section on trade with trade barriers, the amount of climate change was endogenized. However, for a given amount of climate change, the effects of this change can also be endogenous. By adapting to future climate change, a country can become less vulnerable to future variability. Similarly, a country could try to adapt its production possibilities towards the production of goods where the country can be expected to have a comparative advantage in the future. How this would affect the calculations is difficult to say without modeling it. One implication would be that expectations of future climate change would decrease the resources available today, since some of these resources would be used for adapting to future climate change. In terms of the two-country model with trade in goods, this could be modeled with two periods where the productivity in the south in the first period would decrease if there were expectations of climate change. In such a model, the same negative indirect effects for the north would then be present in the first period. In terms of the model with trade in financial instruments, expectations of future climate change would decrease the available endowments in the first period in the countries that want to adapt to future changes. When looking at trade in goods and interpreting changes in productivity as the result of mitigation policy, the exogeneity assumptions may be more reasonable since the time span is shorter.

In this chapter, I used a Ricardian type of model for trade. This type of model is not very well suited to capture the significant volume of trade in similar goods that takes place between similar countries. Furthermore, a share of GDP is not traded at all. So an improvement of the model could be to include a non-traded sector. The countries could then be interpreted as larger regions and the trade in similar goods among similar countries would then belong to the non-traded sector. As argued above, this would probably not change the sign of the indirect effect on countries that are not directly affected by climate change, but it could have a large impact on the net effects in countries that are affected.

This chapter has illustrated two channels through which economies are interconnected and shown that there will be indirect effects in addition to the direct effects experienced by countries. There will likely also be other types of indirect effects of climate change. Commonly discussed examples include migration and increased risks of conflicts. I leave these and other aspects to future research.

In conclusion, I would like to reemphasize the general theme here: since countries are interconnected in many ways, any calculation of the effects of climate change that a country will experience should be based on the total, general equilibrium effects.

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## 4.A Calculations for trade in goods with trading costs

In this appendix, I show the calculations for the two-country model for trade in goods with import tariffs.

### 4.A.1 Comparative statics for changes in $a_1$, $a_2$, $\tau_1$ and $\tau_2$}

Consider a change in $x \in \{a_1, a_2, \tau_1, \tau_2\}$. This change will result in endogenous changes in equilibrium values of $c_{i,j}$. Let primes denote derivatives with respect to $x$ (for each choice of $x$, the corresponding $a_1'$, $a_2'$, $\tau_1'$ or $\tau_2'$ will be one and the other derivatives will be zero) . Differentiating the equilibrium conditions (4.15)-(4.18) with respect to $x$ gives

$$\begin{aligned}
a_1' &= \left(1 + \frac{1}{\sigma}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{1,1}}\right) c_{1,1}' + \frac{\sigma-1}{\sigma}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{1,2}} c_{1,2}' - \frac{l_2}{l_1}\frac{c_{2,1}}{1-\tau_1}\tau_1' \\
a_2' &= \frac{\sigma-1}{\sigma}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{2,1}} c_{2,1}' + \left(1 + \frac{1}{\sigma}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{2,2}}\right) c_{2,2}' - \frac{l_1}{l_2}\frac{c_{1,2}}{1-\tau_2}\tau_2' \\
a_1' &= c_{1,1}' + \frac{l_2}{l_1} c_{2,1}' \\
a_2' &= \frac{l_1}{l_2} c_{1,2}' + c_{2,2}'.
\end{aligned}$$

Setting $x = a_1$, $a_1' = 1$ and $a_2' = \tau_1' = \tau_2' = 0$ gives

$$\begin{aligned}
\frac{\partial c_{1,1}}{\partial a_1} &= \frac{\frac{a_2}{c_{2,2}} + \frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \\
\frac{\partial c_{1,2}}{\partial a_1} &= \frac{\frac{\sigma-1}{\sigma}\frac{c_{1,2}}{c_{1,1}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \\
\frac{\partial c_{2,1}}{\partial a_1} &= \frac{c_{2,1}}{c_{1,1}} \frac{1 + \frac{1}{\sigma}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{2,2}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \\
\frac{\partial c_{2,2}}{\partial a_1} &= \frac{l_1}{l_2} \frac{\frac{1-\sigma}{\sigma}\frac{c_{1,2}}{c_{1,1}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}.
\end{aligned}$$

Setting $a_2' = 1$ and $a_1' = \tau_1' = \tau_2' = 0$ gives

$$\frac{\partial c_{1,1}}{\partial a_2} = \frac{l_2}{l_1} \frac{\frac{1-\sigma}{\sigma}\frac{c_{2,1}}{c_{2,2}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}$$

$$\frac{\partial c_{1,2}}{\partial a_2} = \frac{c_{1,2}}{c_{2,2}} \frac{1 + \frac{1}{\sigma}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{1,1}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}$$

$$\frac{\partial c_{2,1}}{\partial a_2} = \frac{\frac{\sigma-1}{\sigma}\frac{c_{2,1}}{c_{2,2}}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}$$

$$\frac{\partial c_{2,2}}{\partial a_2} = \frac{\frac{a_1}{c_{1,1}} + \frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}.$$

Setting $x = \tau_1$, $\tau_1' = 1$ and $a_1' = a_2' = \tau_2' = 0$ gives

$$\frac{\partial c_{1,1}}{\partial \tau_1} = \frac{l_2}{l_1} \frac{1 + \frac{1}{\sigma}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{2,2}}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_1}$$

$$\frac{\partial c_{1,2}}{\partial \tau_1} = \frac{\frac{1-\sigma}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_1}$$

$$\frac{\partial c_{2,1}}{\partial \tau_1} = -\frac{1 + \frac{1}{\sigma}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{2,2}}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_1}$$

$$\frac{\partial c_{2,2}}{\partial \tau_1} = \frac{l_1}{l_2} \frac{\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_1}.$$

Setting $x = \tau_2$, $\tau_2' = 1$ and $a_1' = a_2' = \tau_1' = 0$ gives

$$\frac{\partial c_{1,1}}{\partial \tau_2} = \frac{l_2}{l_1} \frac{\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_2}$$

$$\frac{\partial c_{1,2}}{\partial \tau_2} = -\frac{1 + \frac{1}{\sigma}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{1,1}}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_2}$$

$$\frac{\partial c_{2,1}}{\partial \tau_2} = \frac{\frac{1-\sigma}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_2}$$

$$\frac{\partial c_{2,2}}{\partial \tau_2} = \frac{l_1}{l_2} \frac{1 + \frac{1}{\sigma}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{1,1}}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_2}.$$

Differentiating utility given by (4.2) with respect to $x$ gives

$$\frac{\partial c_i}{\partial x}u(c_i) = u'(c_i)\frac{\partial}{\partial x},$$

where

$$\frac{\partial c_i}{\partial x} = \left(\frac{c_i}{c_{i,1}}\right)^{\frac{1}{\sigma}}\frac{\partial c_{i,1}}{\partial x} + \left(\frac{c_i}{c_{i,2}}\right)^{\frac{1}{\sigma}}\frac{\partial c_{i,2}}{\partial x}.$$

When calculating this derivative for different $i$ and $x$, I use expressions (4.19) and (4.20). I also use that $\frac{l_2 c_{2,1}}{l_1 c_{1,1}} = \frac{a_1}{c_{1,1}} - 1$ and $\frac{l_1 c_{1,2}}{l_2 c_{2,2}} = \frac{a_2}{c_{2,2}} - 1$.

For changes in $a_1$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial a_1} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}}\frac{\frac{a_2}{c_{2,2}} + \frac{\sigma-1}{\sigma}\left(1 + \left(\frac{a_1}{c_{1,1}} - 1\right)\frac{1}{1-\tau_1}\right)}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}$$

$$\frac{\partial c_2}{\partial a_1} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}}\frac{l_1}{l_2}\frac{c_{1,2}}{c_{1,1}}\frac{\frac{1}{\sigma}\frac{1}{1-\tau_2}\frac{a_2}{c_{2,2}} + \frac{\tau_2}{1-\tau_2}\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}.$$

Note that the second derivative is unambiguously positive since $\frac{a_2}{c_{2,2}} \geq 1 \geq \tau_2$.

For changes in $a_2$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial a_2} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}}\frac{l_2}{l_1}\frac{c_{2,1}}{c_{2,2}}\frac{\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}$$

$$\frac{\partial c_2}{\partial a_2} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}}\frac{\frac{a_1}{c_{1,1}} + \frac{\sigma-1}{\sigma}\left(1 + \left(\frac{a_2}{c_{2,2}} - 1\right)\frac{1}{1-\tau_2}\right)}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}.$$

For changes in $\tau_1$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial \tau_1} = \left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}}\frac{l_2}{l_1}\frac{\frac{1}{\sigma}\frac{a_2}{c_{2,2}} + \frac{1-\sigma}{\sigma}\frac{\tau_1}{1-\tau_1}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}\frac{c_{2,1}}{1-\tau_1}$$

$$\frac{\partial c_2}{\partial \tau_1} = -\left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}}\frac{l_1}{l_2}\frac{\frac{1}{1-\tau_2}\frac{1}{\sigma}\frac{a_2}{c_{2,2}} + \frac{\tau_2}{1-\tau_2}\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}\frac{c_{1,2}}{1-\tau_1}.$$

For changes in $\tau_2$, the changes in aggregate consumption are given by

$$\frac{\partial c_1}{\partial \tau_2} = -\left(\frac{c_1}{c_{1,1}}\right)^{\frac{1}{\sigma}} \frac{l_2}{l_1} \frac{\frac{1}{1-\tau_1}\frac{1}{\sigma}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{2,1}}{1-\tau_2}$$

$$\frac{\partial c_2}{\partial \tau_2} = \left(\frac{c_2}{c_{2,2}}\right)^{\frac{1}{\sigma}} \frac{l_1}{l_2} \frac{\frac{1}{\sigma}\frac{a_1}{c_{1,1}} + \frac{1-\sigma}{\sigma}\frac{\tau_2}{1-\tau_2}}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} \frac{c_{1,2}}{1-\tau_2}.$$

### 4.A.2 Calculations for endogenizing climate

The second-order partial derivatives of $c_1$ are[2]

$$\begin{aligned}
\frac{\partial^2 c_1}{\partial a_2^2} &= \frac{\partial c_1}{\partial a_2}\left[\frac{1}{\sigma}\frac{1}{c_1}\frac{\partial c_1}{\partial a_2} - \frac{1}{\sigma}\frac{1}{c_{1,1}}\frac{\partial c_{1,1}}{\partial a_2} + \frac{1}{c_{2,1}}\frac{\partial c_{2,1}}{\partial a_2} - \frac{1}{c_{2,2}}\frac{\partial c_{2,2}}{\partial a_2}\right] \\
&\quad + \frac{\partial c_1}{\partial a_2}\left[\frac{-\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}^2}\frac{\partial c_{1,1}}{\partial a_2}}{\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}} - \frac{-\frac{a_1}{c_{1,1}^2}\frac{\partial c_{1,1}}{\partial a_2} + \frac{1}{c_{2,2}} - \frac{a_2}{c_{2,2}^2}\frac{\partial c_{2,2}}{\partial a_2}}{\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} + \frac{1}{\sigma}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)}\right] \\
\frac{\partial^2 c_1}{\partial a_2 \partial \tau_1} &= \frac{\partial c_1}{\partial \tau_1}\left[\frac{1}{\sigma}\frac{1}{c_1}\frac{\partial c_1}{\partial a_2} - \frac{1}{\sigma}\frac{1}{c_{1,1}}\frac{\partial c_{1,1}}{\partial a_2} + \frac{\frac{1}{\sigma}\left(\frac{1}{c_{2,2}} - \frac{a_2}{c_{2,2}^2}\frac{\partial c_{2,2}}{\partial a_2}\right)}{\frac{1}{\sigma}\frac{a_2}{c_{2,2}} + \frac{1-\sigma}{\sigma}\frac{\tau_1}{1-\tau_1}}\right] \\
&\quad + \frac{\partial c_1}{\partial \tau_2}\left[-\frac{\frac{1}{\sigma}\left(-\frac{a_1}{c_{1,1}^2}\frac{\partial c_{1,1}}{\partial a_2} + \frac{1}{c_{2,2}} - \frac{a_2}{c_{2,2}^2}\frac{\partial c_{2,2}}{\partial a_2}\right)}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} + \frac{1}{c_{2,1}}\frac{\partial c_{2,1}}{\partial a_2}\right] \\
\frac{\partial^2 c_1}{\partial a_2 \partial \tau_2} &= \frac{\partial c_1}{\partial \tau_2}\left[\frac{1}{\sigma}\frac{1}{c_1}\frac{\partial c_1}{\partial a_2} - \frac{1}{\sigma}\frac{1}{c_{1,1}}\frac{\partial c_{1,1}}{\partial a_2} + \frac{-\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}^2}\frac{\partial c_{1,1}}{\partial a_2}}{\frac{1}{\sigma}\frac{1}{1-\tau_1}\frac{a_1}{c_{1,1}} + \frac{\tau_1}{1-\tau_1}\frac{\sigma-1}{\sigma}}\right] \\
&\quad + \frac{\partial c_1}{\partial \tau_2}\left[-\frac{\frac{1}{\sigma}\left(-\frac{a_1}{c_{1,1}^2}\frac{\partial c_{1,1}}{\partial a_2} + \frac{1}{c_{2,2}} - \frac{a_2}{c_{2,2}^2}\frac{\partial c_{2,2}}{\partial a_2}\right)}{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}}\right) + \frac{1}{\sigma^2}\left(\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} - 1\right)} + \frac{1}{c_{2,1}}\frac{\partial c_{2,1}}{\partial a_2}\right].
\end{aligned}$$

I will analyze the case where the tariffs are small and set $\tau_1 = \tau_2 = 0$. Then, combining (4.13) and (4.14), $\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}} = 1$ and we arrive at

$$\frac{c_{1,1}}{a_1} = 1 - \frac{c_{2,2}}{a_2}$$
$$\frac{a_1}{c_{1,1}} + \frac{a_2}{c_{2,2}} = \frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}.$$

[2] Note that $\frac{c_{2,1}c_{1,2}}{c_{1,1}c_{2,2}}$ only depends on the tariffs and is therefore independent of $a_2$.

The derivatives wrt $a_2$ are

$$\frac{\partial c_{1,1}}{\partial a_2} = \frac{l_2}{l_1}\frac{\frac{1-\sigma}{\sigma}\frac{c_{2,1}}{c_{2,2}}}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}} = \frac{c_{1,1}}{c_{2,2}}\frac{\frac{\sigma-1}{\sigma}\left(\frac{a_1}{c_{1,1}}-1\right)}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}$$
$$\frac{\partial c_{2,1}}{\partial a_2} = \frac{c_{2,1}}{c_{2,2}}\frac{\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}$$
$$\frac{\partial c_{2,2}}{\partial a_2} = \frac{\frac{a_1}{c_{1,1}}+\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}$$
$$\frac{\partial c_1}{\partial a_2} = \frac{c_1}{c_{2,2}}\frac{1}{\sigma}\frac{\frac{a_1}{c_{1,1}}-1}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}.$$

Some useful combinations of derivatives are

$$\frac{1}{c_1}\frac{\partial c_1}{\partial a_2} - \frac{1}{c_{1,1}}\frac{\partial c_{1,1}}{\partial a_2} = \frac{1}{c_{2,2}}\frac{\frac{a_1}{c_{1,1}}-1}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}$$
$$\frac{1}{c_{2,1}}\frac{\partial c_{2,1}}{\partial a_2} - \frac{1}{c_{2,2}}\frac{\frac{a_1}{c_{1,1}}\frac{\partial c_{2,2}}{\partial a_2}+1}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}} = -\frac{1}{c_{2,2}}\frac{\frac{1}{\sigma}\frac{a_2}{c_{2,2}}+\left(\frac{a_1}{c_{1,1}}\right)^2+\frac{a_1}{c_{1,1}}}{\left(\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}\right)^2}$$
$$\frac{1}{c_{2,2}}\frac{\partial c_{2,2}}{\partial a_2} - \frac{1}{c_{1,1}}\frac{\partial c_{1,1}}{\partial a_2} = \frac{1}{c_{2,2}}\frac{\frac{a_1}{c_{1,1}}}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}\frac{2\sigma-1}{\sigma}.$$

Using these, we obtain

$$\frac{\partial^2 c_1}{\partial a_2^2} = \frac{\partial c_1}{\partial a_2}\frac{1}{c_{2,2}}\frac{\frac{1-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2-\frac{1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}\right)^2}$$
$$\frac{\partial^2 c_1}{\partial a_2\partial \tau_1} = \frac{\partial c_1}{\partial \tau_1}\frac{1}{c_{2,2}}\frac{\frac{2-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2+\frac{\sigma-1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}\right)^2}$$
$$\frac{\partial^2 c_1}{\partial a_2\partial \tau_2} = \frac{\partial c_1}{\partial \tau_2}\frac{1}{c_{2,2}}\frac{\frac{1}{\sigma}\left(\frac{a_1}{c_{1,1}}-1\right)+2\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}}+\frac{a_2}{c_{2,2}}}$$

and

$$u''(c_1)\left(\frac{\partial c_1}{\partial a_2}\right)^2 + u'(c_1)\frac{\partial^2 c_1}{\partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial a_2}\frac{1}{c_{2,2}}\frac{\frac{1-\theta-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2 - \frac{1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}\right)^2}$$

$$u''(c_1)\frac{\partial c_1}{\partial \tau_1}\frac{\partial c_1}{\partial a_2} + u'(c_1)\frac{\partial^2 c_1}{\partial \tau_1 \partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial \tau_1}\frac{1}{c_{2,2}}\frac{\frac{2-\theta-\sigma}{\sigma}\left(\frac{a_1}{c_{1,1}}\right)^2 + \frac{\sigma-1}{\sigma}\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}{\left(\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}\right)^2}$$

$$u''(c_1)\frac{\partial c_1}{\partial \tau_2}\frac{\partial c_1}{\partial a_2} + u'(c_1)\frac{\partial^2 c_1}{\partial \tau_2 \partial a_2^2} = u'(c_1)\frac{\partial c_1}{\partial \tau_2}\frac{1}{c_{2,2}}\frac{\frac{1-\theta}{\sigma}\left(\frac{a_1}{c_{1,1}} - 1\right) + 2\frac{\sigma-1}{\sigma}}{\frac{a_1}{c_{1,1}}\frac{a_2}{c_{2,2}}}$$

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