GROWTH, RENEWABLES AND THE OPTIMAL CARBON TAX*

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Abstract

Optimal climate policy is studied in a Ramsey growth model with exhaustible oil reserves, an infinitely elastic supply of renewables, stock-dependent oil extraction costs and convex climate damages. We concentrate on economies with an initial capital stock below that of the steady state of the carbon-free economy and the initial cost of oil (extraction cost plus scarcity rent and social cost of carbon) below that of renewables. There are then two regimes. If the oil stock is small, the social optimum path consists of an initial oil-only phase followed by a renewables-only phase. With a lower cost of renewables or a lower discount rate, more oil is left in situ and renewables are phased in more quickly. The optimal carbon tax rises along its development path during the oil-only phase, but the rise flattens off as less accessible reserves are explored and the social cost of carbon increases. Subsidizing renewables without an optimal carbon tax induces more oil to be left in situ and a quicker phasing in of renewables, but oil is depleted more rapidly (Green Paradox). The net effect on global warming is thus ambiguous. The second regime occurs if the initial oil stock is large. The social optimum then consists of an initial oil-only phase followed with a final oil-renewables phase. This regime also converges to the steady state of the carbon-free economy. The paper also gives a full characterization of two other regimes that occur if the initial capital stock is above its steady state or renewables have an initial cost advantage.

Keywords: Green Ramsey model, carbon tax, renewables, exhaustible resources, global warming, development, growth, intergenerational inequality aversion, second best, Green Paradox

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1. Introduction

Global warming and global economic development and how to reconcile the two are the most pressing issues facing our societies today. A substantial and possibly rising carbon tax is needed not only to curb demand for fossil fuel, but also to accelerate the switch from a carbon-based to a carbon-free economy. It does this, on the one hand, by encouraging the market to leave more oil in the crust of the earth, and, on the other hand, by making renewables more competitive and speeding up the moment of time that they are phased in. Subsidies for carbon-free renewables are, unless there are substantial learning-by-doing or other market failures, a poor substitute for a credible and lasting anticipated path of present and future carbon taxes. The optimal path for the carbon tax depends on the cost of renewables versus the cost of extracting fossil fuel, where the latter will increase as less accessible reserves have to be explored. The optimal carbon tax should be set to the social (external) cost of carbon, which is the present value of all future marginal damages from global warming. But the social cost of carbon is highly endogenous as it depends on both the level of consumption and capital in the economy. More mature economies have a lower marginal utility of consumption and thus a higher social cost of carbon and a higher carbon tax.

Our main objective is to develop a Green Ramsey model, which can be used to analyze the intricate tradeoffs between economic development and fighting global warming with an emphasis on different regimes of energy use (i.e., only oil, only renewables or both). The transition and timing of the transition between these regimes is endogenous. Our model has exhaustible oil[1] reserves, an infinitely elastic supply of renewables, stock-dependent oil extraction costs and convex climate damages. If the initial social cost of oil consisting of the marginal extraction cost, the Hotelling rent and the social cost of carbon is less than that of renewables, the economy starts off using oil. For an economy starting with a capital stock below the steady state of the carbon-free economy, we then formally establish that there are two regimes. The first one occurs if the initial stock of oil is small enough and has an initial oil-only phase followed by a final phase where only renewables are is used. The second regime occurs if the initial stock of oil is large enough and has an initial oil-only phase followed by a final phase where renewables and oil are used simultaneously. In this regime consumption and capital overshoot the steady-state values of the carbon-free economy. In contrast with the first regime, oil use is higher in the early part of the initial oil-only phase and is curbed back substantially in the latter part. In the first regime global warming is primarily curbed in the early part of the initial oil-only regime, whilst in the second regime it is curbed primarily in the latter part. Also, the second oil-abundant regime never phases out oil and phases in renewables at a

[1] From now on we will use for the sake of brevity ‘oil’ rather than ‘fossil fuel’, so oil refers to natural gas as well.

much later date than the first oil-scarce regime does. Furthermore, we establish that the "laissez-faire" outcome always has an initial oil-only phase followed by a carbon-free, renewables-only phase.

A third regime occurs if renewables have an initial cost advantage, which is the case if the cost of renewables is low, the initial oil stock is low and thus oil extraction costs are high, the discount rate is low and initial capital and consumption are high as then the social cost of carbon is high. Renewables are then used forever and oil is never phased in.

Finally, for economies with initial capital stocks above the carbon-free steady-state value, we establish three possibilities. The third regime of using renewables forever is enlarged for all initial oil stocks below the steady state. If the initial oil stock is above the steady state but not too much, a fourth regime occurs where the economy sets off using only renewables and then switches at some time to using renewables and oil alongside each other. If the initial oil stock is quite a bit above steady state, the range in which the second regime, where the initial oil-only phase is followed with a phase with simultaneous use of oil and renewables, occurs is enlarged.

Until there is a breakthrough in renewables technology, the third and fourth regimes will not occur. So our discussion focuses mostly on the first and second regimes where oil has an initial cost advantage.

The innovation of our paper is not only the precise characterization of which of the various regimes takes place and whether oil and renewables are used on their own or used simultaneously in the production process, but also the endogenous determination of the optimal switch time between the different phases of each regime and the optimal amount of oil to be left unused in the crust of the earth.

A recent paper by Golosov et al. (2011) is in some respects close to ours. It looks at backstops in a Ramsey or Dasgupta-Heal-Solow-Stiglitz type model with stock-dependent extraction costs, albeit that they model global warming externalities as directly impacting aggregate production whereas we allow for them as losses to social welfare. Their main findings are that a constant carbon tax rate does not affect oil use at all; the time path of the optimal carbon tax has an inverse U-shape, where the eventual decline of the carbon tax results from their assumption of natural decay of the concentration of CO2 in the atmosphere and that oil reserves are fully exhausted. For their analytical results they make five assumptions: (i) a logarithmic utility function; (ii) a Cobb-Douglas production function; (iii) a negative exponential function for multiplicative output damages; (iv) 100 per cent depreciation of manmade capital at the end of each discrete period; and (v) zero extraction costs for oil. These strong assumptions deliver a closed-form expression for the value function and a constant ratio of consumption to output. As a result, the social cost of carbon is proportional to output and is high if the climate-sensitivity parameter is large and the social rate of discount is small.

Our interests in characterizing the optimal climate policy are threefold: first, what are the determinants of the date that renewables kick in and the date that oil is phased out altogether; second, are renewables going to be used on their own or alongside oil and what is the optimal sequencing of oil and renewables; and third, how much oil is it socially optimal to leave in situ. We thus devote considerable attention to the intricate connections between the oil economy and the carbon-free economy. In the oil-scarce economy, the optimal carbon tax rises during the phase where oil is used until the moment that renewables replace oil. The rise in the optimal carbon tax flattens with time, since private agents hold back depletion as less accessible reserves have to be explored. Furthermore, the size of the optimal carbon tax depends on the stage of economic development of the economy. If the economy is at a low level of economic development, it has a low carbon tax rate which rises as the economy develops which echoes the prescriptions of those who argue in favor of a rising ramp for the carbon tax (e.g., Nordhaus, 2007). However, a more mature economy has a higher carbon tax rate. In a mature enough economy, the economy can switch to an oil-abundant regime with overshooting of consumption and capital. In that case, the optimal carbon tax eventually starts to fall.

The switch to renewables occurs more quickly in a socially optimum than in a market outcome which does not internalize global warming damages. The amount of oil reserves that is left in situ in the optimal outcome is determined by the condition which says that the cost advantage of the last drop of oil over the backstop exactly equals the present value of marginal global warming damages. We show that more oil reserves are left in situ if the social rate of discount is low, the climate challenge is acute, and the initial stock of oil reserves is high. The stock that is left in situ is higher if the economy is more developed and consumption is relatively close to its steady state. Conversely, if the economy is still underdeveloped and consumption low, less oil reserves will be left in situ.

We thus extend the classic Ramsey model of economic growth to allow for natural exhaustible resources, renewable backstops and global warming damages. Our analysis is related to the famous DHSS growth model with investment in manmade capital and natural exhaustible resources as factors of production (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974), but differs in that we allow for renewable backstops and global warming damages. Our analysis is also related to earlier studies on pollution in the Ramsey model (e.g., van der Ploeg and Withagen, 1991), on capital accumulation, oil depletion and backstops (e.g., Tahvonen, 1997; Tsur and Zemel, 2003, 2005), on pollution and climate change in models with depletion of exhaustible resources but without investment and growth (e.g., Krautkraemer, 1985, 1998; Withagen, 1994), and on those studying regimes with simultaneous use of oil and renewables albeit in economies without capital (Hoel and Kverndokk, 1996).

Our analysis extends earlier results on the second-best effects of subsidizing clean backstops on oil exhaustion, speed and duration of phasing in of the backstop, and the effects on green and total welfare (Hoel, 2008; van der Ploeg and Withagen, 2010; Grafton, Kompas and Long, 2010; Gerlagh, 2011) to allow for saving, investment and capital accumulation. We thus offer a general-equilibrium analysis of the Green Paradox (Sinn, 2008ab), which says that subsidizing clean renewables (e.g., solar or wind energy) instead of implementing a carbon tax is counterproductive as owners of oil fields are encouraged to pump more quickly and global warming is accelerated. However, if the social discount rate that is used is low (cf., the Stern Review (2007)) and global warming damages are acute, it is optimal to leave more oil reserves in the crust of the earth in which case cheaper renewables induce a bigger fraction of oil reserves to remain unexploited and global warming damages fall (van der Ploeg and Withagen, 2010). The Green Paradox then does not occur. We show, within the context of our Green Ramsey model, that this paradox is less likely to occur for a mature than a developing economy with relatively scarce capital, high marginal utility of consumption and low marginal global warming damages. This dilemma for the early stages of economic development reminds one of Bertolt Brecht's dictum: 'Erst kommt das Fressen: Dann kommt die Moral' (from *Die Dreigroschenoper)*.

Section 2 presents the Green Ramsey model and the optimality conditions for the social optimum. Section 3 discusses as a preliminary the carbon-free economy and the so-called threshold economy, which is defined as the economy where after an initial oil-only phase the steady state of the carbon-free economy is exactly reached and then the economy stays there forever after. Section 4 discusses the regimes that can occur if oil has an initial cost advantage. If the initial stocks of oil and capital are low enough, the initial oil-only phase is followed by a renewables-only phase. If the initial stocks are below the threshold value, the initial oil-only phase is followed by a final phase where oil and renewables are used simultaneously. Section 4 also analyzes how the optimal remaining stock of oil and the optimal time of the switch to using renewables only or using both renewables and oil are determined. Section 5 first shows that a rising carbon tax rate ensures that the market outcome yields the socially optimal outcome and that the optimal carbon tax rate is lower in the earlier than in the later stages of development. It then discusses second-best climate policy for the market economy if an optimal carbon tax is infeasible and why this is more likely to lead to a Green Paradox if the economy is still developing. Section 6 presents simulations to illustrate the time path of the optimal carbon tax for the oil-abundant regime of the Green Ramsey model. It also shows how this path depends on the stage of economic development, the aversion to intergenerational inequality and the rate of time preference, and sheds light on the Green Paradox. Section 7 offers some policy simulations for the oil-abundant regime which has a final phase where oil and renewables are used simultaneously. Section 8 characterizes all the other regimes that can occur if renewables have an initial cost advantage and/or the economy starts off with a capital stock above steady state. Section 9 concludes.

2. The Green Ramsey model

Let $O(t) \geq 0$ denotes oil use and $S(t) \geq 0$ the stock of remaining oil reserves at instant of time t. Then along a feasible program we have for all $t \geq 0$:

(1) $$\dot{S}(t) = -O(t), \quad S(0) = S_0,$$

where $S_0 \geq 0$ is the given initial stock of oil reserves. Hence, total oil depletion cannot exceed initial reserves: $\int_0^\infty O(t)dt \leq S_0$. We abstract from natural degradation of CO2 in the atmosphere, so the stock of atmospheric carbon E equals the initial stock plus the accumulated sum of past CO2 emissions:

(2) $$E(t) = E_0 + \int_0^t O(t)dt = E_0 + S_0 - S(t),$$

where we have normalized so that the CO2-emission ratio equals one.

Manmade capital K and energy are inputs in the production process, which is described by the production function F. We suppose that energy from oil, O, and energy from renewables, R, are perfect substitutes.[2] We denote by $G(S)$ the cost needed to extract one unit of oil.

Assumption 1: $0 < G(S_0) < b, G'(S) < 0, \forall S > 0, \lim_{S \to 0} G'(S) = \infty, \lim_{S \to \infty} G'(S) = 0.$

We assume that the cost of extraction one unit of oil rises as fewer oil reserves are left, $G'(S) < 0$, and that oil extraction costs become infinitely large as oil reserves become fully exhausted. The latter assumption ensures that oil reserves will never be fully exhausted. The cost of renewables $b > 0$ is higher than the initial cost of extracting oil, $b > G(S_0)$, so that the market finds it attractive to start with using oil.

The material balance equation of the economy and the investment dynamics are:

(3) $$\dot{K} = F(K, O+R) - bR - G(S)O - C - \delta K, \quad K(0) = K_0,$$

where C is consumption, δ the depreciation rate of manmade capital, and K_0 the initial capital stock.

[2] Wind and solar energy are more likely to be complements than substitutes for oil. We assume perfect substitutes for analytical convenience; relaxing it would introduce regimes where the two types of fuel are used alongside each other. See Smulders and van der Werf (2008) and Michielsen (2011) for partial equilibrium studies which do allow for imperfect substitution between oil and the backstop in the partial equilibrium analysis of climate policy.

Assumption 2: F has non-increasing returns to scale. It is strictly concave and increasing on $\mathbb{R}^2_{++}$. $F(K,0) = F(0,O+R) = 0$. $\lim_{O+R\to 0} F_{O+R}(K,O+R) = \infty$, $\lim_{O+R\to\infty} F_{O+R}(K,O+R) = 0$ for all $K > 0$.[3]

$\lim_{K\to 0} F_K(K,O+R) = \infty$, $\lim_{K\to\infty} F_K(K,O+R) = 0$ for all $O+R > 0$.

The production function thus satisfies the Inada conditions. Assumption 2, together with $\delta > 0$, implies that without the use of the backstop the economy cannot maintain a positive constant level of consumption (Dasgupta and Heal , 1974).

Intertemporal social welfare W depends on utility of consumption and damage from accumulated CO2:

(4) $$W = \int_0^\infty e^{-\rho t} \left[U(C(t)) - D(E_0 + S_0 - S(t)) \right] dt,$$

where $\rho > 0$ is the constant social rate of discount. The instantaneous utility function, U, is concave and satisfies the Inada condition, which ensures positive consumption throughout. Global warming damages, D, and marginal damages increase in the stock of atmospheric carbon[4].

Assumption 3: $U'(C) > 0, U''(C) < 0, \forall C > 0. \lim_{C\to 0} U'(C) = \infty.\ \ D'(E) > 0, D''(E) > 0, \forall E > 0.$

The social planner maximizes (4) subject to (1)-(3) and the non-negativity constraints. Defining the social price of energy as $p = F_{O+R}(K,O+R)$, we obtain the following proposition.

Proposition 1: The social optimum follows from (1)-(3) and the optimality conditions:

(5) $$p \leq b,\ R \geq 0, \text{c.s.},$$

(6) $$\dot{p} = r\left[p - G(S) \right] - \frac{D'(E)}{U'(C)} \text{ if } O > 0,$$

(7) $$\dot{C}/C = \sigma \left[F_K(K,O+R) - \delta - \rho \right],$$

and the transversality condition, where $r \equiv F_K - \delta$ is the net rate of return on capital and $\sigma \equiv -U'(C)/CU''(C) > 0$ the elasticity of intertemporal substitution.

Proof of proposition 1: The problem is to maximize (4) subject to (1), $\dot{E} = O$ and (3). Letting μ_K be the marginal social value of manmade capital, μ_S the marginal social value of oil reserves and μ_E the

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[3] For the derivative of the production function with respect to energy, we use F_O, F_R and F_{O+R} interchangeably.

[4] Temperature is a concave function of accumulated CO2 emissions. So even if damages are a convex function of temperature, damages need not necessarily be a convex function of accumulated CO2 emissions. An alternative is to model global warming as damaging production as in the RICE and DICE models of Nordhaus (2007).

marginal social value of a *clean* environment, the Hamiltonian function for this problem reads $H \equiv U(C) - D(E) + \mu_K \;\; F(K, O+R) - C - bR - G(S)O - \delta K \;\; - \mu_S O - \mu_E O.$ We thus obtain the necessary optimality conditions:

(8a) $\quad U'(C) = \mu_K, \quad \rho\mu_K - \dot{\mu}_K = \;\; F_K(K, O+R) - \delta \;\; \mu_K,$

(8b) $\quad F_R(K, O+R) \le b$ and $R \ge 0,$ c.s.,

(8c) $\quad F_O(K, O+R) - G(S) - (\mu_S + \mu_E) / \mu_K \le 0$ and $O \ge 0,$ c.s.,

(8d) $\quad \rho\mu_S - \dot{\mu}_S = -G'(S)O\mu_K, \qquad \rho\mu_E - \dot{\mu}_E = D'(E),$

(8e) $\quad \lim_{t\to\infty} \;\; \mu_K(t)K(t) + \mu_S(t)S(t) - \mu_E(t)E(t) \;\; e^{-\rho t} = 0.$

Bearing in mind $E = E_0 + S_0 - S$, we define $\mu = \mu_S + \mu_E$. If $R > 0$, $p = F_R = b$ which gives (5). If $O > 0$, (8c) gives $p = G(S) + \mu / \mu_K$ and thus (8c) together with (8a) and (8e) give (6) where the first part of (8d) gives the marginal value of the oil stock as the discounted value of all future lower extraction costs and the second part of (8e) gives the social cost of the CO2 stock (the social cost of carbon) as the discounted value of all future marginal damages. Equation (7) follows from (8a). The transversality condition (8e) or $\lim_{t\to\infty} \;\; \mu_K(t)K(t) + \mu(t)S(t) \;\; e^{-\rho t} = 0$ amounts to $\lim_{t\to\infty} \mu(t)e^{-\rho t} = 0,$ since $\mu_K(t), K(t), S(t)$ converge to constants as $t \to \infty.$ Q.E.D.

Equation (5) states that the marginal product of the renewable backstop, if in use, equals its marginal cost *b*; if it is not used, its marginal product is less than its cost. If oil is used in the production process, (6) indicates that the social price of oil follows a modified version of the Hotelling rule. The return on leaving a marginal barrel of oil in the earth (i.e., the capital gains on oil reserves) should thus equal the return from depleting a marginal barrel of oil, selling it and obtaining a market rate of return on it (i.e., the return on capital net of the cost of extraction) *minus* the marginal global warming damages resulting from burning this barrel of oil (after converting these from utility to final goods units). Strictly speaking, this only has a market interpretation if there is a carbon tax which is set to the social cost of carbon as discussed in section 5.1. Without global warming externalities and extraction costs, we get the familiar Hotelling rule which says that the capital gains on oil should equal the market rate of interest. Note that the Hotelling rent on oil $p - G(S)$ vanishes when the "laissez-faire" economy relying on only oil approaches the moment in time where the renewable backstop is introduced (cf., Heal, 1976). Equation (6) can also be written as $\dfrac{d \;\; p - G(S)}{dt} = r \;\; p - G(S) \;\; + G'(S)O - \dfrac{D'(E)}{U'(C)},$ which says that the Hotelling rent on oil increases at a lower rate than the market rate of interest for two reasons. First, extracting more oil pushes up extraction costs, which makes oil depletion more conservative. Second, extracting more oil raises the stock of atmospheric carbon and this pushes up marginal climate damages, which makes oil depletion also more conservative.

Equation (7) is the familiar Keynes-Ramsey rule for consumption growth. Growth in consumption is high if the return on capital is high and consumers are relatively patient, especially so if there is not much intergenerational inequality aversion. The coefficients of relative intergenerational consumption inequality aversion and relative risk aversion equal $1/\sigma$. Since the marginal utility of consumption is set to the social value of capital, $U'(C)=\mu_K$, consumption decreases in the social value of capital.

For future reference, we note that the marginal utility of consumption is set to the social value of capital, μ_K, the social value of the stock of oil in the earth equals μ_S, and the social cost of the stock of carbon in the atmosphere equals μ_E. Furthermore, if oil is in use, the marginal social cost of oil consists of three components. The first is the unit extraction cost. The second is the scarcity rent on oil, μ_S/μ_K, which is the discounted value of all future benefits in terms of lower extraction costs of having a higher oil stock. Indeed, it follows from (8b) that $\mu_S(t)=-\int_t^{\infty}e^{-\rho(s-t)}G'(S(s))O(s)\lambda(s)ds$. The third component is the marginal social value of a *clean* environment, μ_E/μ_K (in terms of resource units). Putting these components together, we have that the social price is set to the marginal social cost of oil which consists of the oil extraction cost plus the scarcity rent of oil plus the social cost of carbon, $p=G(S)+(\mu_S+\mu_E)/\mu_K$. Following Nordhaus (2011), the social cost of carbon (*SCC*) is defined as the reduction in the capital stock resulting from keeping an extra marginal unit of carbon in the soil rather than in the atmosphere, evaluated along the optimum path (so $dW = 0$ with welfare to go given by $W=W(K,S,E)$:

$$\text{(9)} \qquad SCC \equiv -\left.\frac{dK}{dE}\right|_W = p-G(S)-\frac{\mu_S}{\lambda}=\frac{\mu_E}{\lambda}=\frac{\int_t^{\infty}e^{-\rho(s-t)}D'(E(s))ds}{U'(C(t))}>0,$$

which corresponds to the present discounted value of marginal global warming damages (in goods units).

3. Preliminaries: the carbon-free economy, the oil-only phase and the threshold economy

Before we characterize the regimes of our Green Ramsey model and the optimal ordering and existence of the various phases of energy use, we describe the carbon-free economy and the equations describing the phase where only oil is used in the production process. We then introduce the hypothetical 'threshold' economy, which is defined as an economy which starts off with using only oil in the production process until a time T^* when the economy reaches the steady-state of the carbon-free economy indicated by (K^*, C^*) and stays there forever after.

3.1. The carbon-free economy

By definition the carbon-free economy uses no oil: O = 0. Demand for renewables then follows from $F_R(K,R)=b.$ Due to assumption 2 this yields R as a smooth function that is increasing in the stock of manmade capital and decreasing in the cost of renewables:

(10) $R=V(K,b),\quad$ with $V_K=-F_{KV}/F_{VV}>0$ and $V_b=1/F_{VV}<0.$

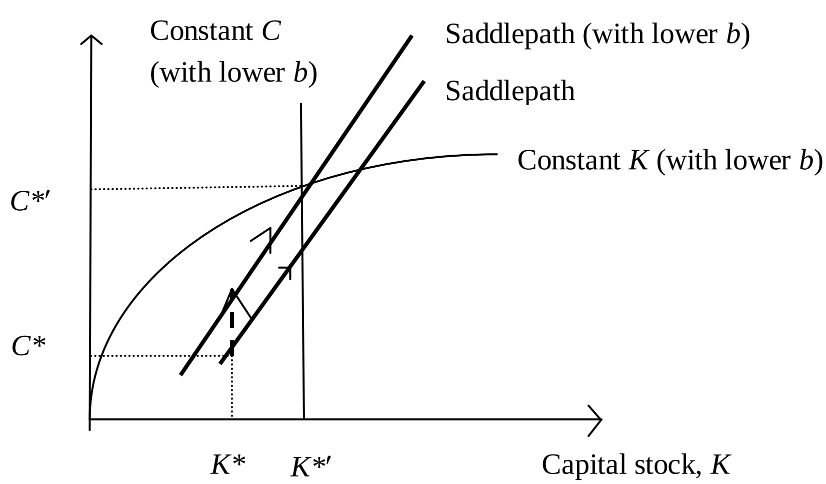
Defining output net of fuel costs as $\tilde{F}(K,b)\equiv F(K,V(K,b))-bV(K,b),$ where $\tilde{F}_K=F_K>0$ and $\tilde{F}_b=-V(K,b)<0,$ we write the material balance equation and the Keynes-Ramsey rule as:

(3X) $\dot{K}=\tilde{F}(K,b)-\delta K-C,$

(7X) $\dot{C}/C=\sigma\left[\tilde{F}_K(K,b)-\delta-\rho\right]$.

Assumption 2 implies that there exists a unique interior steady state of the carbon-free economy denoted by (K^*, C^*, R^*) and defined by $F_K(K^*,R^*)=\rho+\delta,\ F_R(K^*,R^*)=b$ and $C^*=F(K^*,R^*)-bR^*-\delta K^*.$

Figure 1: Impact of a lower cost of renewables in the carbon-free economy



From any initial stock of capital, the system converges towards the steady state. Fig. 1 shows that a lower resource cost b leads to an upward jump in consumption; afterwards consumption and capital increase along the saddlepath $C(t) = \Theta^R\left(K(t); b\right)$, where we have included the backstop cost as a parameter. The suffix R indicates the carbon-free economy. The steady state is on the saddlepath: $C^* = \Theta^R(K^*; b)$.

3.2. What initial oil stock ensures that the economy stays at the steady state?

The value of $S(T) = S^*$ which ensures that some point in time T, the economy holds the steady-state capital stock, $K(T) = K^*$, and some oil stock $S(T)$, and is indifferent between using oil and renewables is such that the cost of renewables, b, equals the marginal cost of using oil (i.e., oil extraction costs $G(S^*)$ plus the total discounted marginal damage of using oil now and no longer in the future, $D'(E_0 + S_0 - S^*) / \rho U'(C^*))$:[5]

$$(11) \qquad b = G(S^*) + \frac{D'(E_0 + S_0 - S^*)}{\rho U'\left(\Theta^R(K^*; b)\right)}.$$

If (11) holds and $S(T) = S^*$, the economy uses renewables forever and never phases in oil because using an extra unit of oil would raise the extraction cost and the social cost of carbon .

3.3. The only-oil phase

If the economy starts off using only oil in the production process (i.e., R = 0), it is described by (1),

$$(3O) \qquad \dot{K} = F(K, O) - G(S)O - C - \delta K, \quad K(0) = K_0,$$

$$(6O) \qquad \dot{p} = \left(F_K(K, O) - \delta\right)\left(p - G(S)\right) - \frac{D'(E_0 + S_0 - S)}{U'(C)},$$

$$(7O) \qquad \dot{C} = \sigma\left(F_K(K, O) - \delta - \rho\right)C.$$

Suppose that, at some instant of time T, renewables are phased in alongside oil or instead of oil. Then it is clear that the final oil price is pinned down by the cost of renewables. Hence, given T, S_0 , K_0 , $p(T) = b$ and some $C(T)$, we can solve the system of ODEs for $p(0)$, $C(0)$, $K(T)$, and $S(T)$.

3.4. The threshold economy

Now consider the 'threshold' economy: at the end of the oil-only phase at time T^* the economy has reached the steady state of the carbon-free economy and stays there forever after. Let us take the initial

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[5] Alternatively, consider (5) and (6). With simultaneous use of oil and renewables, the economy is indifferent between the two sources of energy and p is constant, so $b = G(S) + D'(E_0 + S_0 - S) / (F_K - \delta)U'(C)$ with $F_K - \delta = \rho$ if the economy happens to be in the carbon-free steady state. This also gives (11).

capital stock as given and assume $K_0 < K^*$ and that it is optimal to start with only oil.[6] We also suppose that the initial oil stock is large enough, so that there exists a unique initial oil stock that ensures that the economy at the end of the oil-only phase exactly reaches the steady state of the carbon-free economy. We denote this threshold value by S_0^*.[7] To determine S_0^* and the associated switch time T^*, we first define S^* as in (10) but now with S_0^* still to be determined:

$$(12) \qquad 0 < S^* = \Upsilon(\overset{-}{b}, \overset{+}{C^*}, E_0 \overset{+}{+} S_0^*, \overset{-}{\rho}) < S_0 \text{ from } G(S^*) + \frac{D'(E_0 + S_0^* - S^*)}{\rho U'(C^*)} = b,$$

The stock of oil to be left in situ at the switch to the carbon-free economy, $S(T^*) = S^*$, follows from the condition that the marginal cost of extracting the last drop of oil, $G(S^*)$, plus the associated social cost of carbon, $D'(E_0 + S_0 - S^*) / \rho U'(C^*)$, must equal the cost of renewables, *b*. At time T^* the social cost of carbon is the present value of all future marginal global warming damages from time T^*, i.e., $SCC = \dfrac{D'(E_0 + S_0^* - S)}{\rho U'(C^*)}$. Fig. 2 shows the social cost of carbon (*SCC*) as a decreasing function of remaining oil reserves and the extraction cost advantage of oil over renewables $(MCA = b - G(S(T^*)))$ as an increasing function of the remaining stock of oil reserves.[8] The equilibrium stock of oil reserves follows from intersection of the *MCA* and *SCC* curves.

To determine the threshold value of the initial stock of oil reserves, S_0^*, and the associated switch time, T^*, we need to solve (1), (3O), (6O) and (7O) given the five boundary conditions, $S_0 = S_0^*$, $K(0) = K_0$, $C(T^*) = C^*, K(T^*) = K^*$, $p(T^*) = b$ and (12). We must ensure that $F_O(K, O) < b$ holds throughout this initial oil-only phase, which is the case as the social price of oil *p* rises monotonically to *b* at time T^*. Given $K(0) = K_0$ and *b*, the five boundary conditions can be solved for the following five unknowns: $S_0 = S_0^*$, $C(0) = C_0^*$, $p(0) = p_0^*$, $S(T^*) = S^*$ and T^*. We restrict attention to those values of K_0 for which a solution satisfying $S_0^* \geq 0$ and $T^* \geq 0$ exists.

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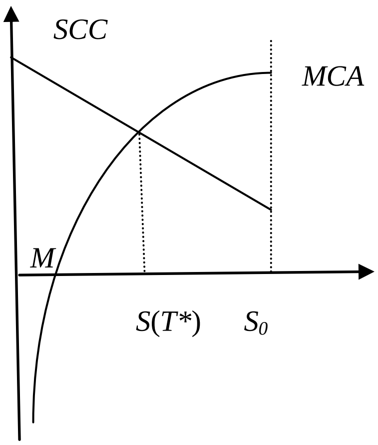
[6] This requires that the initial oil stock is not too small and the initial stock of atmospheric carbon is not too high; else, it is optimal to start with using only renewables. See section 8 and appendix 3.

[7] If the initial oil extraction cost plus initial social cost of carbon is less than the cost of renewables, i.e., $G(S_0) + D'(E_0) / \rho U' \;\; \Theta^R(K_0, b) \;\; > b$, then it never pays to use oil and thus no value of S_0^* exists if S_0 is too small.

[8] If we do not assume that oil extraction costs become infinitely large as reserves vanish, the *MCA* may start off higher than the *SCC* in which case there would be full exhaustion of oil reserves.

Figure 2: How much oil to be left in the crust of the earth?



A lower cost of renewables or rate of discount boosts C^*, curbs $U'(C^*)$ and further raises the cost of carbon. The downward shift in the *MCA* curve (in case of a lower *b)* and the upward shift in the SCC curve in fig. 2 imply that, for a given S_0^*, the stock of oil to be left in situ is pushed up. It also leads to a higher steady-state value of consumption, which lowers the marginal utility of consumption and thus also pushes up the stock of oil to be left in situ. It thus follows that a higher value of S_0^* is required. These changes will also bring forward the date that renewables are phased in. Hence, the threshold value S_0^* decreases and the switch time T^* increases in *b* and ρ.

The threshold value of S_0^* depends negatively on the initial stock of capital as a lower K_0 requires a higher S_0^* and is associated with a higher switch time T^*.[9] A lower K_0 also depresses initial consumption $C(0) = C_0^*$ and the initial social price of oil $p(0) = p_0^*$ in the threshold economy.

These arguments give rise to the following proposition.

Proposition 2: The critical value for the initial oil stock and, the associated switch time and initial consumption and social price of oil in the threshold economy satisfy:

$$(13) \qquad S_0^* = S_0^*(\overset{-}{K_0}, \overset{-}{b}, \overset{-}{\rho}), \quad T^* = T^*(\overset{(-)}{K_0}, \overset{+}{b}, \overset{+}{\rho}), \quad C_0^* = C_0^*(\overset{+}{K_0}, \overset{-}{b}, \overset{+}{\rho}), \quad p_0^* = p_0^*(\overset{+}{K_0}, \overset{+}{b}, \overset{-}{\rho}) \le b,$$

where the − and + signs indicate partial derivatives (in brackets the likely sign).

[9] With more capital, the economy uses more oil, produces more and reaches the carbon-free steady state more quickly. However, the economy will also consume more and save less, so that capital will grow less quickly and this tends to postpone the date at which K^* is reached. Computations with the parameter values used in sections 6 and 7 suggest that the former effect dominates the latter effect.

Hence, the threshold corresponds to a critical value for the initial oil stock S_0^* given K_0 or equivalently to a critical value for the initial capital stock $K_0^* = K_0^*(\bar{S}_0, \bar{b}, \bar{\rho})$ given S_0.

4. Optimal solution paths of the Green Ramsey model: oil-scarce and oil-abundant regimes

In this section we formally characterize the optimal sequencing and phases of the socially optimal outcome and discuss the optimal amount of oil to be left in the crust. We focus attention at the situation where the economy starts off with a stock of manmade capital stock below the steady state of the carbon-free economy and oil has an initial social cost advantage over renewables.

Assumption 4: $G(S_0) + \dfrac{D'(E_0)}{\rho U'\ \Theta^R(K_0;b)} < b$ and $K_0 < K^*$.

It is only optimal to start using oil if initial oil extraction cost plus initial social cost of carbon is less than the cost of renewables, i.e., $G(S_0) + \dfrac{D'(E_0)}{\rho U'\ \Theta^R(K_0;b)} < b,$ where $\Theta^R(K_0;b)$ is the initial rate of consumption that leads a carbon-free economy to its steady state.[10] [11] Hence, to guarantee that the economy starts off with using oil, oil must be abundant enough so that the initial cost of oil including the cost of carbon is below the cost of renewables. Taking also account of the downward impact on the social cost of carbon and the social price of oil of a higher discount rate and a lower initial stock of carbon in the atmosphere E_0, we see that the condition in the first part of assumption 2 is more easily satisfied and thus that the oil stock can be somewhat lower and it is still optimal to start with oil. Similarly, a high level of the initial capital stock implies a high rate of initial consumption, a low marginal utility of consumption and thus a high social cost of carbon in terms of production units. The critical value of the oil stock that ensures that it is attractive to start with oil is thus lower. We summarize our condition by rewriting the condition for it being optimal to start with oil (i.e., the first part of assumption 2) as follows:

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[10] If $G(S_0) + D'(E_0)/\rho U'\ \Theta^R(K_0;b)\ \geq b,$ then take $O(t) = 0,\ \mu(t) = D'(E_0)/\rho,\ \mu_K(t) = U'(C(t))$ with $C(t)$ the optimal carbon-free, monotonically increasing consumption rate. It is easily verified that $G(S_0) + \mu(t)/\mu_K(t) > b$ for all $t > 0.$ Moreover, the proposed program satisfies all necessary conditions and is thus optimal. Hence, if $G(S_0) + D'(E_0)/\rho U'\ \Theta^R(K_0;b)\ \geq b,$ it is optimal never to use oil. To have an interesting problem, we assume $G(S_0) + D'(E_0)/\rho U'\ \Theta^R(K_0;b)\ < b,$ so that it is optimal to start with using only oil. We exclude the possibility of starting with simultaneous use in appendix 1.

[11] As already stated, we suppose that (for a given initial oil of capital) the initial oil stock is not so small that it is optimal to start with using renewables from the outset. If the economy did start using renewables from the outset, it would use them for ever after as the relative scarcity of oil (as indicated by $p - G(S)$) increases as the economy grows towards its carbon-free steady state.

(14) $S_0 > S_0^{**}(\bar{K}_0, \bar{b}, \bar{\rho})$.

Given assumption 2, we establish in proposition 4 below the optimal phases of the Green Ramsey model for two regimes. In regime I initial stocks of oil and capital are small enough (i.e., $S_0 < S_0^{*}(\bar{K}_0, \bar{b}, \bar{\rho})$ from proposition 2) so that it is optimal to follow the initial oil-only phase with a final renewables-only phase provided initial extraction cost plus initial cost of carbon does not exceed the cost of renewables (i.e., provided (14) holds and the initial oil stock is not too small). In regime II initial stocks of oil and capital are so large (i.e., $S_0 > S_0^{*}(\bar{K}_0, \bar{b}, \bar{\rho})$) that the initial oil phase must be followed with a regime where oil and renewables are used simultaneously.

Given that the economy starts off with using only oil, extraction costs and marginal climate damages increase as more oil is depleted and carbon is emitted into the atmosphere. Furthermore, along a development path rising consumption leads to falling marginal utility of consumption and rising marginal climate damages in production units and thus also leads to rising time profile of the social cost of carbon. Hence, there must be a point in time at which the social cost of oil has become as big as the cost of renewables. At that point renewables are phased in either alongside oil or instead of oil. From then on it is never optimal to start using only oil for ever again. For if that were the case, consumption eventually goes to zero as oil reserves are exhaustible whilst a positive level of consumption can be attained using the infinite supply of the backstop. Hence, it is suboptimal never to phase in renewables as ever-decreasing levels of consumption with asymptotic depletion of oil reserves violate the Inada condition.

Proposition 3: An oil-only regime is never interrupted by a period where only renewables are used. Furthermore, any oil-only regime eventually comes to an end.
Proof: For the first part, see appendix 1. The second part has been discussed above.

Proposition 4 below establishes that once renewables have been phased in, given that $K_0 < K^*$, it is never attractive to have a phase using only oil in the production process again. Proposition 4 establishes the precise conditions for the two regimes that can occur in this economy using the critical value for S_0, i.e., S_0^*, and the corresponding transition time, T^*, characterized in proposition 2.

Proposition 4: If $S_0 = S_0^*$, the initial only-oil phase reaches (K^*, C^*, S^*) at time T^* and from then on only renewables are used forever after. Given assumption 4, there are two regimes in both of which it is optimal to start with using only oil:

I. If $S_0 < S_0^*$, renewables take over from oil before the carbon-free steady state is reached. Simultaneous use of oil and the backstop does not occur. Capital and consumption increase monotonically towards the carbon-free steady state. The switch time $T > 0$ is such that the oil left in situ satisfies $S(T) = \Upsilon(\overset{-}{b}, \overset{+}{C(T)}, E_0 \overset{+}{+} S_0, \overset{-}{\rho}) < S^*$.

II. If $S_0 > S_0^*$, consumption and capital overshoot their steady-state values during the initial oil-only phase. After the initial oil-only phase, there is from some time T onwards a final phase where oil and renewables are used alongside each other. During this final phase, oil use declines asymptotically to zero and renewables use rises gradually to its carbon-free steady state. The economy converges asymptotically to the carbon-free steady state.

$\left(K^*, C^*, S^* = \Upsilon(b, C^*, E_0 + S_0, \rho) \right)$. There is no renewables-only phase.

Proof: see appendix 1.

The oil-scarce regime I occurs if the initial oil stock and initial capital stock are sufficiently small (if $S_0 < S_0^*(\overset{-}{K_0}, \overset{-}{b}, \overset{-}{\rho})$). Oil is then eventually replaced by renewables. Also, renewables are never used alongside oil, so that overshooting of capital and consumption does not occur.[12] Typically, renewables are phased in earlier than in the threshold economy, $T < T^*$ (see simulations reported in section 6). The oil-abundant regime II occurs if the initial oil stock is above its threshold (i.e., $S_0 > S_0^*(\overset{-}{K_0}, \overset{-}{b}, \overset{-}{\rho})$), so that the economy is less constrained by the scarcity of oil and thus consumption and capital eventually overshoot their carbon-free, steady-state values. There will be a final phase where consumption and the capital stock fall over time *and* oil and renewables are used alongside each other. A falling oil stock implies a rising oil extraction cost and a rising social cost of carbon so that falling consumption and capital and the resulting rise in the social cost of carbon rise ensure that the condition for simultaneous use, $b = G(S) + \dfrac{D'(E_0 + S_0 - S)}{\left[\tilde{F}_K(K,b) - \delta \right] U'(C)}$, can hold through time.[13]

We know from proposition 2 that a lower cost of renewables b or rate of discount ρ increases the threshold S_0^* and reduces the likelihood that the economy finds itself in the regime starting with a high

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[12] We can see this from the condition for simultaneous use, $b = G(S) + D'(E_0 + S_0 - S) / \left[\tilde{F}_K(K,b) - \delta \right] U'(C)$, which shows that rising C and K must be associated with rising S which is infeasible.

[13] Simultaneous use cannot occur if capital and consumption are growing over time as this would require negative oil use, which is infeasible in the absence of storage.

enough initial oil or capital stock to have an oil-only phase followed by an oil-renewables phase. The corresponding optimal duration of the oil-only regime, T^*, then typically shortens as it will be optimal to phase in renewables more quickly. Furthermore, renewables are phased in later than in the threshold economy, $T > T^*$, and after capital has peaked (see simulations reported in section 7).

We have used a threshold value of S_0^* in proposition 4, but proposition 2 has established that this decreases in K_0. Regime I of proposition 4 thus applies if initial stocks of oil and manmade capital are relatively small compared with the threshold given in proposition 2, but the initial stock of oil is not so small that the initial oil extraction cost plus initial social cost of carbon is higher than the cost of renewables and it does not pay to start with oil. The economy then undershoots and the initial oil-only phase must be followed by a renewables-only phase forever after. Regime II applies if initial oil and capital stocks are larger than the threshold given in proposition 2. The economy then overshoots and the initial oil-only phase must be followed by phase where oil and renewables are used alongside each other forever after. Regime II with simultaneous use of oil and renewables is thus more likely to occur for an oil-rich economy with a high stock of capital.

The conditions for pasting the two phases of regimes I and II and calculating the optimal amount of oil to be left in situ are: (i) the price of energy must be continuous at the moment that renewables are phased in, i.e., $p(T) = b,$ so that there are no unexploited opportunities for improving social welfare; (ii) the paths of capital, consumption and the stock of oil reserves must also be continuous and thus should not jump at the switch time T; (iii) the social cost of the last drop of oil consisting of the oil extraction cost plus the social cost of carbon must equal the cost of renewables as at time T the economy is indifferent between using oil and renewables. Appendix 2 gives further details of the way the solution trajectories are computed.

5. Climate policy, the market and the Green Paradox

5.1. Realizing the socially optimal outcome in the market economy

The market economy which does not internalize global warming externalities ($D \equiv 0$). In such a "laissez-faire" economy there cannot be use of both oil and renewables in the production process: simultaneous use requires $G(S) = b$ if $D = 0$ which gives a constant stock of oil which is inconsistent with positive oil use. Hence, after the initial oil phase, renewables take over forever.

Proposition 5: The market economy without taxes or subsidies never has simultaneous use of oil and renewables. The sequence of use is: first only oil is used for some interval $[0,T]$, and then renewables

take over indefinitely. The amount of oil to be left in situ follows from $b = G(S(t)), t \geq T$ and is less than in the socially optimal outcome. The economy converges asymptotically to the carbon-free steady state.
Proof: see appendix 1.

The next proposition characterizes the optimal carbon tax.

Proposition 6: The government can reproduce the social optimum outcome by levying a carbon tax τ during the fossil-fuel phase $[0,T]$ equal to the social cost of carbon:

$$(15) \qquad \tau(t) = \int_t^\infty \left[\frac{D'(E_0 + S_0 - S(s))}{U'(C(s))} \right] e^{-\int_t^s r(v)dv} ds.$$

The optimal carbon tax rises, falls or stays constant over time in the oil-only phase if $r\tau$ is greater than, smaller than, or equal to $\frac{D'(E)}{U'(C)}$.

Proof: Appendix 1 establishes that indeed $\tau(t) = \mu_E(t) / \mu_K(t)$.

Since there are no other distortions in the economy apart from the climate externality and lump-sum taxes/subsidies are available, the optimal carbon tax must equal the social cost of carbon where the latter is defined as the present discounted value of all future marginal global warming damages using the market rate of interest and not the rate of time preference.

If the economy is in a low stage of development, consumption and the capital stock are low, the interest rate is high and the marginal value of consumption is high so that the social cost of carbon and the optimal carbon tax are low. Hence, as the economy develops and consumption and capital rise, the social cost of carbon and optimal carbon tax rise. Once renewables kick in, the optimal carbon tax stays constant at the level that prevails at the end of the oil-only phase. Hence, the magnitude of the optimal carbon tax depends on the state of economic development. However, if the economy starts off at a high enough degree of economic development, it may be optimal for the carbon tax to start off high and then diminish with time. [14] Indeed, proposition 6 indicates that, if consumption and capital overshoot their steady-state values, the interest rate is relatively small and the marginal utility of consumption is small, there is a real possibility that the optimal carbon tax falls with time, especially if the optimal carbon tax is low and the atmospheric stock of CO2 is high.

5.2. Second-best outcome if a carbon tax is infeasible

[14] If there is natural decay of CO2 in the atmosphere, it is also be optimal for the carbon tax rate to first rise, then to stabilize and finally to fall (cf., Golosov et al., 2010).

What happens if for political or other reasons it is infeasible to levy a carbon tax?[15] The Green Paradox states that subsidizing the backstop fuel with the aim of curbing oil demand and carbon emissions is counterproductive as it encourages private well owners to pump up their oil more rapidly, thereby aggravating global warming damages. This paradox has been studied before in a partial equilibrium model without capital accumulation (Sinn, 2008ab; Hoel, 2008; Gerlagh, 2011; Grafton et al., 2010), but it relies on a fixed supply of oil reserves being fully exhausted. If oil extraction costs rise rapidly as oil reserves diminish, the market leaves less oil in the crust of the earth than the social optimum. In that case, the Green Paradox need not occur as reducing the cost of renewables then leaves more oil in situ and curbs global warming damages (van der Ploeg and Withagen, 2010).

We extend earlier analysis of the Green Paradox to a general equilibrium framework with economic growth. The Green Paradox highlights the second-best effects of introducing a constant backstop subsidy υ financed by a lump-sum tax, to phase out oil more quickly and mitigate global warming if a carbon tax is ruled out, $\tau = 0$. We are interested in the effects of a renewables subsidy on consumption, accumulation of capital, growth and economic development. There is no case for a subsidy (or tax) on renewables once the extraction cost of oil is larger than the production cost of renewables. However, we suppose that the government is able to commit and keeps the renewables subsidy in place once oil is no longer used. This is necessary, because if it is known that the government will remove a renewables subsidy and some oil is left in the ground the private sector might react differently. In the following we introduce a "small" subsidy or tax into the "laissez-faire" economy and then see whether or not this brings us closer to the social optimum. By "small" we mean that the sign of $b - \upsilon - G(0)$ is not reversed.

For the carbon-free (renewables-only) phase we still have equation (3R) and the Euler equation for the carbon-free phase becomes $\dot{C}/C = \sigma\left[\tilde{F}_K(K, b-\upsilon) - \delta - \rho\right]$. We thus see that with a renewables subsidy ($\upsilon > 0$) the economy converges to a steady state with higher capital and consumption. Introduction of a subsidy in an economy that is already carbon free leads to an initial downward jump in the rate of consumption.

In the market economy the sequence of energy use is that the oil-only phase is followed by the renewables-only, carbon-free phase. For a small enough initial stock of capital, overshooting does not occur and the same sequence will occur in the social optimum. The economy without the subsidy on renewables leaves some oil reserves unexploited, but less than in the social optimum. Now, a subsidy has

[15] Of course, in practice, politicians do charge a price for carbon (witness also the European Union Emission Trading Scheme) but lower than the social optimum. For simplicity, we suppose that the government of our global economy levies no carbon tax at all.

the effect of leaving more oil unexploited and brings the economy closer to the first best.[16] The subsidy is beneficial for green welfare (no Green Paradox). Still, aggregate welfare is likely to be lower than in the social optimum, because the distorting effect of the subsidy will dominate the effect on green welfare, since the marginal damage in the case at hand is low. But fine tuning of the subsidy might be cumbersome. We conclude that, if the economy is in the early stage of economic development, the Green Paradox does not necessarily occur as with low levels of consumption and high marginal utility of consumption the valuation of global warming damages is low. So a second-best renewables subsidy makes more sense if the economy is in the early stages of development. As we have seen already, a first-best carbon tax is then lower as well.

6. Policy simulations: oil-scarce regime (undershooting)

To start we offer some policy simulations for the case where the initial conditions are such that the social optimum has an initial period where only oil is used and a final period where only renewables are used (i.e., regime I of proposition 4 with an initial stock oil stock below the threshold of proposition 2). For a given initial oil stock, this occurs for a sufficiently low initial capital stock. We simulate both the social optimum and the decentralized market outcome (described in sections 4 and 5). Normalizing so that $S_0 = 20$, we set $E_0 = 24$.[17] W use the discount rate, $\rho = 0.014.$ We use the CES utility function $U(C) = C^{1-1/\sigma} / (1-1/\sigma)$ with a ballpark estimate of the elasticity of intertemporal substitution equal to $\sigma = 0.5$[18] and explore the sensitivity with respect to σ to gain insight into the effect of intergenerational inequality aversion on global warming and economic growth. We use a Cobb Douglas production function $F(K, O+R)) = K^{\alpha}(O+R)^{\beta}$. The shares of labor and of oil/gas in GDP have been set at $\alpha = 0.2$ and $\beta = 0.1.$ The average lifetime of manmade capital has been set at twenty years, so $\delta = 0.05.$

[16] If full exhaustion of oil reserves is feasible (i.e., if oil extraction costs as oil reserves vanish are a small enough finite number rather than infinity), a backstop subsidy only leads to more rapid pumping of oil, faster exhaustion of oil reserves and thus to a Green Paradox (van der Ploeg and Withagen, 2010).

[17] In 2000 there were oil and gas reserves in the crust of the earth corresponding to 469 and 1,121 Giga tons of carbon, respectively, whereas there had been emitted 224 plus 111 Giga tons of carbon into the atmosphere resulting from burning, respectively, oil and gas (Edenhofer and Kalkuhl, 2009). Normalizing so that $S_0 = 20$, we set $E_0 = 24$ (rather than 335x20/1,590 = 4.2) to allow for the substantial CO2 concentration that was already in the atmosphere for non-anthropogenic reasons.

[18] Some argue that the elasticity of intertemporal substitution σ is very low with an implied coefficient of relative risk aversion of about 10 (e.g., Mehra and Prescott, 1985; Campbell and Mankiw, 1989; Obstfeld, 1994); others argue that σ is one or greater than one with a much smaller and more realistic implied coefficient of relative risk aversion (e.g., Hansen and Singleton, 1982). $\sigma = 0.5$ implies a coefficient of relative risk aversion 2 which seems a little high. Rather than breaking the link between risk aversion and intertemporal substitution to allow for an elasticity of intertemporal substitution in the range 0.05 to 1 and a coefficient of relative risk aversion in the range 0.4 to 1.4 (Epstein and Zin, 1991), we will explore the sensitivity of our results with respect to different values of σ.

The initial stock of capital is set at half the value that prevails in the steady state of the carbon-free economy, i.e., $K_0 = K^* / 2$. We set $b = 0.5$[19] and the initial cost of extracting one unit of oil at $G(S_0) = 0.2$. We suppose that unit extraction costs become infinitely large as more and more oil is extracted and capture this with the specification $G(S) = \gamma S_0 / S$ with $\gamma = 0.2$. This implies that in the market outcome where global warming externalities are not internalized, half of the initial stock of oil is left in situ at the time of the switch to the renewable backstop, $S(T) = S_0 / 2 = 10$. If global warming externalities are internalized, a bigger stock is left in situ. The cost of extracting the last drop of oil thus exactly equals the cost of renewables in the market outcome, but will be less than in the socially optimal outcome. For global warming damages we use the specification $D(E) = \kappa E^2 / 2$ with $\kappa = 0.00012$.[20] With these parameters, we calculate $S_0^* = 20.8$ and $T^* = 24.3$. Since we start with $S_0 = 20 < S_0^*$, proposition 4 indicates that it is optimal to start with an initial oil-only phase and end with a final renewables-only phase. Appendix 2 gives details of the algorithm we have used to simulate our model.

6.1. The optimal carbon tax needed to attain the first-best outcome in the market economy

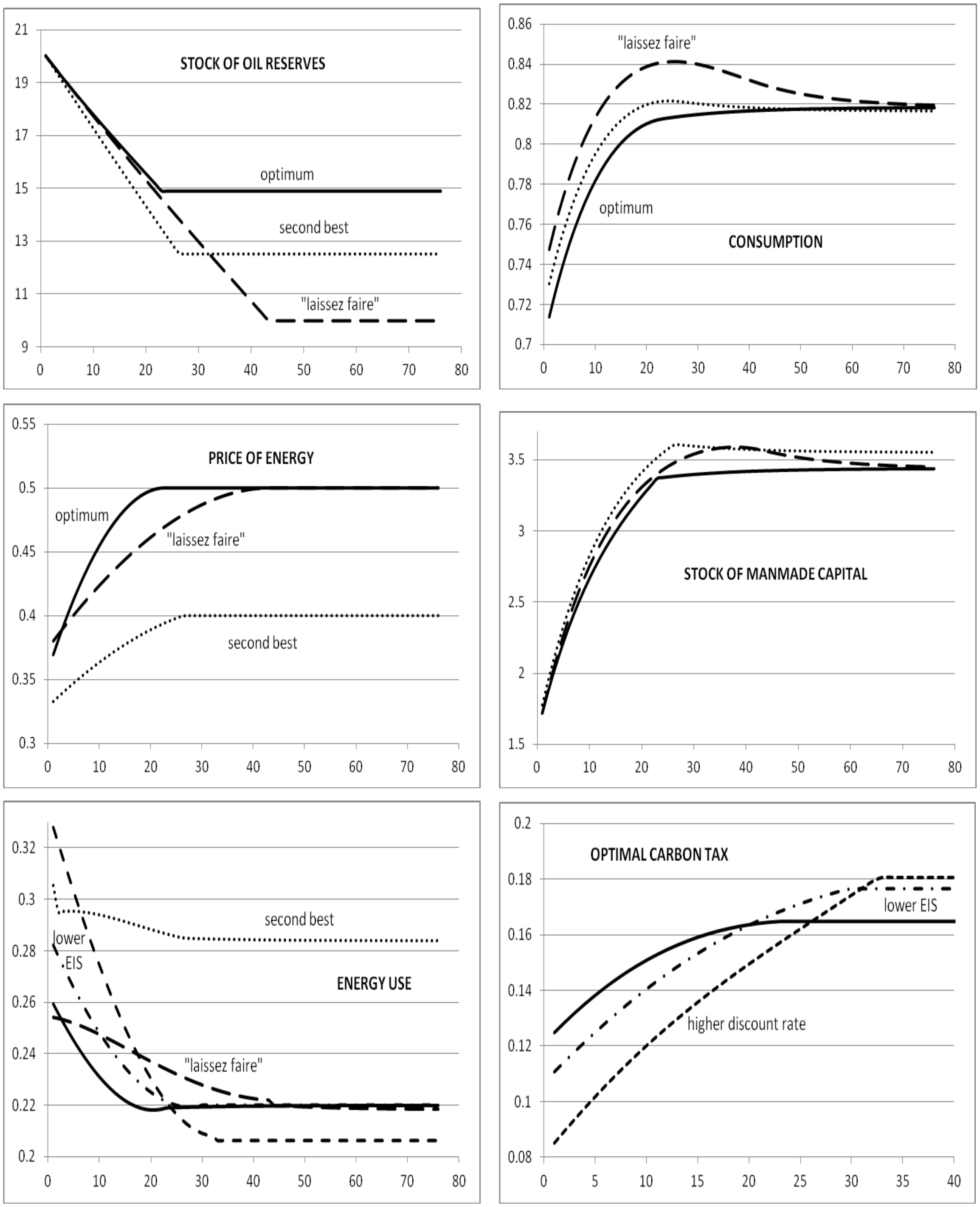
Fig. 3 plots the time paths of the key variables for these two outcomes (optimal solid lines, market long-dashes) and also gives the time path of the optimal carbon tax that ensures that the market properly internalizes the global warming externality. Both manmade capital and consumption rise over the entire optimal path, confirming the theory. In this benchmark simulation there is no overshooting of capital in the optimum, but the market does overshoot. Under “laissez faire”, manmade capital and consumption decline during the carbon-free phase and even before. Optimally internalizing global warming externalities implies that renewables get phased in more quickly than the social optimum, namely at time 22.0 rather than 42.3[21], and that 14.9 rather than 10.0 units of oil are left in situ at the time of the switch to the carbon-free economy. Switching more quickly to the carbon-free economy and leaving more oil in situ is an effective way to curb CO2 emissions and global warming. Consumption and manmade capital

[19] Solar and wind energy are more expensive than fossil fuel, especially if one looks beyond marginal production costs once capacity is installed and considers the costs needed to increase capacity, deal with intermittence and repair offshore wind mills. Wind energy can be at least three times as expensive as ‘grey’ electricity (Wikipedia). As far as the electricity industry is concerned, costs of renewables have fallen substantially: solar energy is currently 50% more expensive than conventional electricity; wind energy has the same cost and is (apart from the problem of intermittence) competitive; and biomass, CCS coal/gas and advanced natural gas combined cycle have mark-ups of 10%, 60% and 20%, respectively (Paltsev et al., 2009). These mark-ups are measured from a very low base and may not be so impressive when they account for a much larger market share. Hence, we use a 100% mark-up.

[20] Peer-reviewed estimates of the social cost of carbon for 2005 have an average of $43 per ton of carbon and a standard deviation of $83 dollar per ton of carbon, and these estimates are likely to increase by 2 to 4 percent per year (Yohe et al., 2007). A ballpark estimate of the social cost of carbon is $30 dollar per ton (Nordhaus, 2007).

[21] The switch occurs earlier than in the threshold economy, $T = 22.0 < T^* = 24.3$, as oil is scarcer.

Figure 3: Simulation trajectories for oil-scarce economy



Key: solid lines – optimum (benchmark): $T = 22.0,\ S(T) = 14.9$;
long dashes – "laissez faire": $T = 42.3,\ S(T) = 10.0$;
dots – no carbon tax and backstop subsidy (second best): $T = 25.3,\ S(T) = 12.5$;
dots and dashes – optimum with higher inequality aversion: $T = 28.9,\ S(T) = 13.1$;
short dashes – optimum with higher rate of time preference: $T = 31.8,\ S(T) = 12.0$.

are smaller for the optimal than for the "laissez-faire" outcome, because the use of oil is cut back more quickly to limit global warming. We also observe a steeper time path for the price of oil in the socially optimal outcome due to the rising time profile for the optimal carbon tax from 0.24 to 0.31. This contrasts with the inverse U-shape for the time profile of the optimal carbon tax found in Golosov et al. (2011), where the eventual decline of the carbon tax results from their assumption of natural decay of CO2 in the atmosphere.

The Green Solow model put forward by Brock and Taylor (2010) has CO2 emissions as an inevitable by-product of production and abstracts from renewables. Social welfare is maximized by choosing constant savings rate and constant share of abatement in output. They find an Environmental Kuznets Curve: emissions initially increase and later decrease with economic development. Within our Green Ramsey framework CO2 emissions per unit of output are initially high, since initially the marginal utility of consumption is large compared to the marginal damages of accumulated CO2. The rapid accumulation of manmade capital compensates the falling use of oil resulting from the rising price of oil and growth tapering off as the economy develops. Consequently, CO2 emissions are initially high and then fall rapidly over time. Once the economy has switched to a clean backstop, CO2 emissions are reduced to zero and the accumulated pollution in the atmosphere is stabilized. So, we also get an Environmental Kuznets Curve in a different framework with a benevolent policy maker.

6.2. Effects of intergenerational inequality aversion and time preference

If the elasticity of intergenerational inequality aversion ($1/\sigma$) is increased from 2 in the benchmark to 4 (corresponding to halving the elasticity of intertemporal substitution ($EIS = \sigma$), we find that the time to phase out oil and switch to renewables in the social optimum is postponed from instant 22.0 to 28.9, and that, as a result of more aggressive oil use, the stock of oil that is left in situ at the end of the oil phase is decreased from 14.9 to 13.1. Both these factors tend to increase CO2 emissions and global warming, as may be expected if intergenerational inequality aversion is higher and thus more priority is given to current, relatively poor generations who have to shoulder most of the burden of combating climate change rather than to distant, relatively rich generations. This way the economy develops faster initially at the expense of global warming, albeit that the steady-state levels of the capital stock and consumption are not affected by more intergenerational inequality aversion. Fig. 3 indicates that, with a lower *EIS*, the optimal carbon tax rate for this case (dots and dashes) is lower in the initial part of the oil-only phase but higher in the latter part of this phase. Furthermore, as the oil-only phase lasts longer, oil use is higher and lasts longer than if intergenerational equality aversion is not so high. With a higher intergenerational equality aversion, current generations are better off in terms of consumption than future generations.

One might argue that the private sector employs a higher rate of time preference than the government, say a rate of time preference of 0.03 rather than 0.014 for the “laissez-faire” economy. The time of the switch towards renewables is then reduced by a tiny amount from 42.34 to 42.30 whilst the stock of oil that remains in situ remains 10.0. However, as the economy is impatient and consumes more upfront and thus invests less, it ends up in the long run with much less manmade capital (2.58 rather than 3.57) and somewhat lower consumption (0.80 rather than 0.83).[22] If the government also employs the higher rate of discount of 0.03, it initially pursues a less aggressive climate change policy resulting in much more oil use. In the latter part of the oil-only phase climate policy becomes less aggressive and oil use is below that if the government employs the prudent discount rate. Still, renewables are phased in more quickly than under “laissez faire” at instant 31.8, but a lot more slowly than if the government employs a precautionary discount rate of 0.014 (at time instant 22.0). Furthermore, oil left in situ, 12.0, is less than with a prudent discount rate of 0.014, but more than in the “laissez-faire” outcome. Fig. 3 indicates that the optimal carbon tax for the case of a low discount rate (solid line) is for the most part lower than that for a high discount rate (short dashes) but in the final part of the oil-only phase is higher.

6.3. Second-best outcome: subsidizing renewables does not lead to the Green Paradox

Fig. 3 also plots the time paths of the key macroeconomic and resource variables under the renewables subsidy (dotted lines) amounting to $\upsilon = 0.1$ and financed with lump-sum taxes. The date of switching from oil to renewables becomes 25.3, later than in the socially optimal outcome and earlier than in the market without the subsidy. The amount of oil left in situ increases from 10 in the “laissez-faire” economy to 12.5, which is less than in the socially optimal outcome. The time path for consumption is higher than in the social optimum, but lower than in the “laissez-faire” market economy. As a result of renewables subsidy, the price of energy is much lower both during the oil-only and the carbon-free phase. Private agents are encouraged to use much more oil in production than even in the “laissez-faire” outcome. This is what underpins the inexorable logic of the Green Paradox: despite renewables being phased in more quickly and more oil being left in situ, private agents pump up oil much more vigorously. For our numerical example the present value of global warming damages is reduced from 2.19 in the “laissez-faire” market outcome to 1.98 (more than in the social optimum, 1.73). Hence, despite the Green Paradox of pumping up more oil, global warming damages need not increase under the backstop subsidy as renewables are phased in more quickly and more oil is left in situ. As the renewables subsidy distorts private decisions, private welfare falls from -67.8 in the “laissez-faire” to -69.0 in the market outcome

[22] The reason that C^* changes only a little compared with K^* is that the share of capital is much smaller than the combined share of all the non-energy factors (capital and labor) in value added.

with the subsidy. The renewables subsidy thus boosts green welfare, but curbs social welfare from -70.0 to -70.9. Clearly, such a subsidy also performs worse than the outcome with an optimal carbon tax.

7. Policy simulations: oil-abundant regime (overshooting)

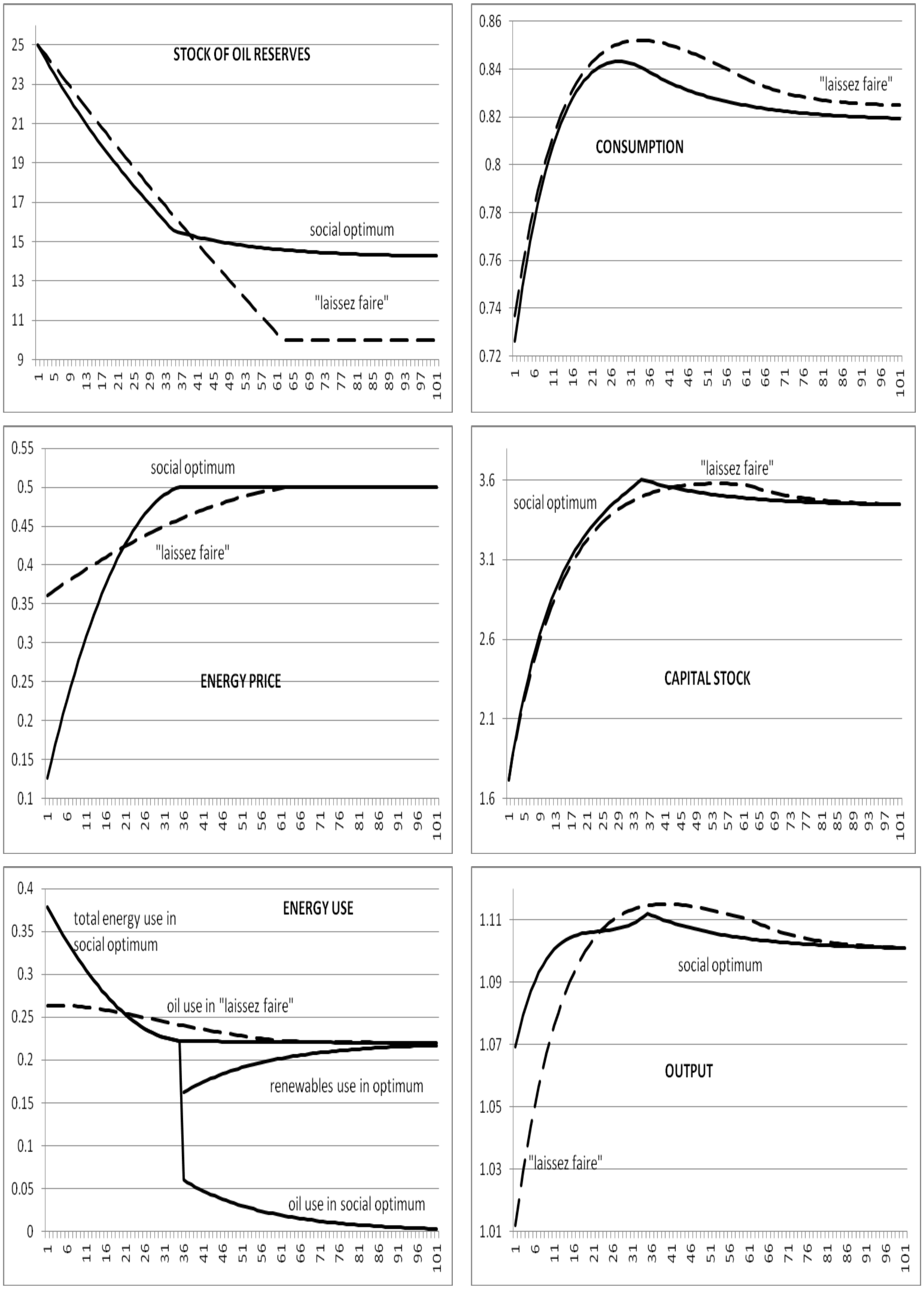
We now offer some policy simulations for regime II of proposition 4 in which the oil-only phase is followed by a final phase where oil and renewables are used simultaneously. In contrast, in the final phase of the "laissez faire" outcome only renewables are used. We use the same parameters as in section 6 except that we set the initial oil stock equal to $S_0 = 25 > S_0^* = 20.8$ instead of 20. The policy simulations of this oil-abundant regime are depicted in fig. 4.

In contrast with the case of oil scarcity, it is now optimal to extract more oil initially than in the market economy. The aim is not to increase consumption but to build up capital more rapidly than in the market economy. The social price of energy in the social optimum starts below that under "laissez faire" and then during the latter part of the oil-only phase is above it. As a result, oil use is only lower than the "laissez-faire" oil use in the latter part of the only-oil phase. As soon as renewables take over under "laissez faire", the market price has caught up with the social price of oil again. Still, renewables fall ever so slightly from then on as capital falls during this final phase under "laissez faire". We also observe that *clean* renewables are phased in much later under "laissez faire" (i.e., at time 61.4 instead of 34.0), albeit that the social optimum never phases out oil completely and only gradually ramps up the use of renewables. Output overshoots in both outcomes. The swinging time profile of output in the social optimum reflects, on the one hand, the rapid growth of the capital stock during the early parts of the initial oil-only phase, and, on the other hand, the substantial curbing of oil use during the oil-only phase. The reason for the kink in net investments and output is that at the transition, capital, consumption and energy use are continuous, but oil and renewables are not. In fact, $bR + G(S)O = b(O+R) + [G(S) - b]O$ is discontinuous at the transition because $[G(S) - b] = \mu / \mu_K \neq 0$. Oil use jumps down at the moment renewables are phased in.

The social optimum achieves an improvement in green welfare at the expense of private welfare by substantially curbing oil use and carbon emissions and thus flattening output growth during the latter part of the oil-only phase, despite the increase in oil use during the early part of this phase. Consumption under "laissez faire" is higher throughout, especially during the latter part of the oil-only phase. The social optimum also leaves much more oil in situ and thus puts less carbon in the atmosphere.

The optimal carbon tax is depicted in fig. 5(a). It continues to rise after renewables have been phased in, since during this final phase capital and consumption fall so that the interest rate and marginal utility of consumption rise. This combined with the increase in marginal climate damages as more carbon is

Figure 4: Simulation trajectories for oil-abundant economy

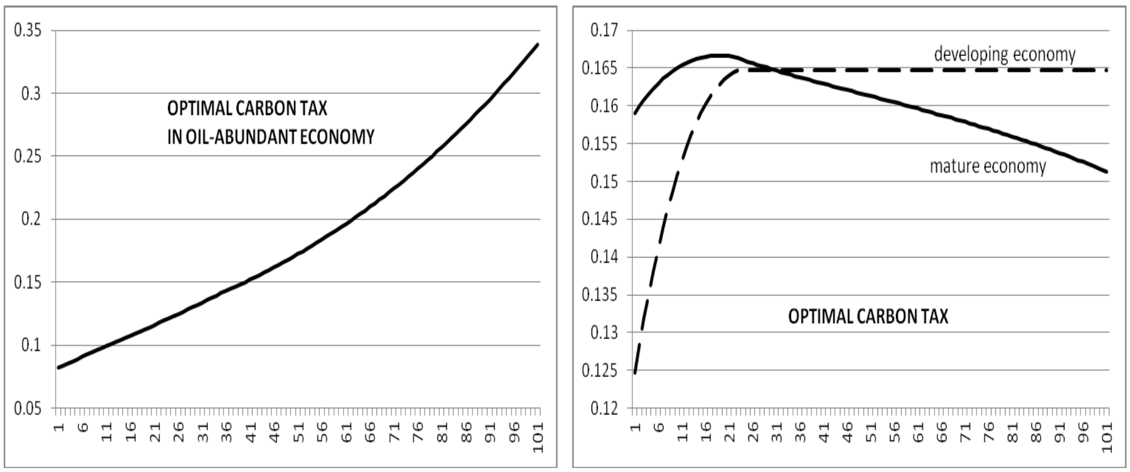


Note: Social optimum $T = 34.0$, $S(T) = 15.6$; $S(\infty) = 14.2$; "Laissez faire" $T = 61.4$, $S(T) = S(\infty) = 10$.

emitted into the atmosphere leads to an upward time profile of the optimal carbon tax in the final phase. During this final phase use of oil in the production process is gradually winded down whilst that of renewables is gradually ramped up.

Figure 5: The optimal carbon tax in the oil-abundant and the mature economy

(a) $K_0 = 0.5K^*, S_0 = 25 > S_0^* = 20.8$ (b) $K_0 = 0.75K^*, S_0 = 20 > S_0^* = 18.7$



If we would start with a larger initial capital stock, say $K_0 = 0.75K^*$ and keep $S_0 = 20$, the threshold for the initial oil stock will be lower, i.e., $S_0^* = 18.7 < 20.8$, which moves the economy also from an oil-scarce to an oil-abundant regime and thus oil is never phased out and at renewables are eventually phased in alongside oil (see appendix 3). Fig. 5(b) indicates that the time profile of the optimal carbon tax in the mature economy has an inverted U-shape. It starts off higher but ends up lower than in the developing economy discussed in section 6. The social cost of carbon for the mature economy is initially higher due to the lower interest rate and the lower marginal utility of consumption. In spite of the fact that there is no decay of the CO2 stock, the carbon tax eventually decreases.

8. Full characterization of all potential regimes of the Green Ramsey model[23]

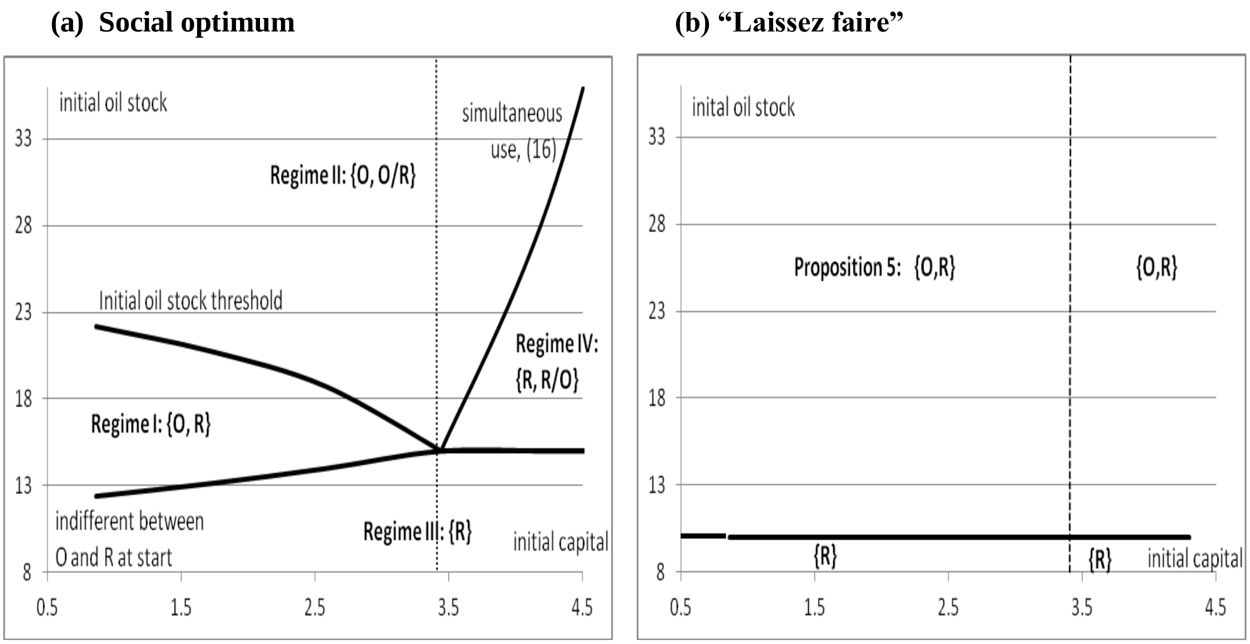
We now extend proposition 4 to give a full characterization of the other socially optimal regimes which occur if assumption 4 does not hold. These regimes are displayed in fig. 6(a) for the parameter values used in sections 6 and 7, but its qualitative shape holds for other parameter values as well. Fig. 6(a) takes all parameters of the model, including the initial carbon stock, fixed, except for the initial stocks of capital and oil. The downward-sloping threshold of proposition 2 shows the critical value of S_0^* as a decreasing function of K_0 ending at K^*. We have used assumption 4 and (14) to also plot the upward-sloping locus

[23] More details of the full characterization of the socially optimal regimes are given in appendix 4.

$S_0 = S_0^{**}(\bar{K}_0, \bar{b}, \bar{\rho})$ above which it is optimal to start with oil and below to start with renewables.[24] Both these loci are only relevant for if the economy starts off with an initial capital stock below its steady state, $K_0 < K^*$, i.e., to the left of the dotted line. This part of fig. 6(a) thus gives the range of values of K_0 and S_0 for which regimes I and II delineated by proposition 4 prevail.

It also gives regime III in the bottom part of fig. 6(a) which applies if it is optimal to start with renewables rather than oil, i.e., if $S_0 < S_0^{**}(\bar{K}_0, \bar{b}, \bar{\rho})$. In that case, oil is so scarce that extraction cost, the scarcity rent and the social cost of carbon (9) are so high that the social cost of oil is never below that of renewables. Alternatively, the initial capital stock is so high enough for that the demand for oil to be so high that it makes oil uncompetitive compared with renewables. Hence, oil is never phased in and renewables are used forever. [25] This regime of using renewables throughout is also more likely to occur if renewables are very cheap and the discount rate is very low.

Figure 6: Characterization of regimes of the Green Ramsey model



[24] As initial capital and thus initial consumption go to zero, the Inada condition on the utility function implies that marginal utility goes to infinity and thus that the second term in (14), i.e., the social cost of carbon, goes to zero, provided μ_E does not go to infinity (which is the case as with given finite S_0, one cannot accumulate an infinite amount of atmospheric CO2). Hence, as initial capital goes to zero, the critical value for the initial oil stock in (14) becomes $S_0^{**}(0, b, \rho) = G^{-1}(b)$. In other words, the lines for which the economy is indifferent between oil and renewables crosses the vertical axis at the critical value of the initial oil stock for the "laissez-faire" economy.

[25] Starting off with renewables on a path of growing capital implies that the social cost of carbon rises all the time and thus oil becomes more and more unattractive and will never take over from renewables. Hence, renewables will never take over. It follows that simultaneous use must require $K_0 > K^*$ (see appendix 1).

Now consider the part of fig. 6(a) where it is still optimal to start with oil but $K_0 > K^*$, i.e., the part to the right of the dotted line. If the initial capital stock exceeds its steady-state value, it is never optimal to have regime I start with only oil and ending with a renewables-only phase. However, regime III where renewables are used forever is also extended to the range $K_0 > K^*$, provided the initial oil stock stays below its steady state, $S_0 < S^*$ (see appendix 4).

To delineate the other two regimes that are optimal to the right of the dotted line and with $S_0 > S^*$, we need to consider the condition under which it is optimal to use oil and renewables alongside each other (condition (A2) in appendix 2). This gives the upward-sloping locus for simultaneous use depicted in fig. 6(a). From (8a)-(8b), this condition amounts to $G(S) + \mu / \mu_K = b$, where we define $\mu \equiv \mu_E + \mu_S$ as the total marginal shadow cost of oil. The scarcity of oil relative to capital is denoted by μ / μ_K. Above the simultaneous-use locus and to the right of the dotted line, the scarcity of oil relative to capital, μ / μ_K, is lower than is necessary for simultaneous use and it is thus optimal to start with oil and extend the regime II of proposition 4, i.e., to start with only oil before using oil and renewables simultaneously (see the top regime in fig. 6(a)). However, below the simultaneous-use locus and to the right of the dotted line, the scarcity of oil relative to capital is bigger than is necessary for simultaneous use and it is optimal to start with renewables. This yields regime IV for high values of the initial capital stock and an intermediate range of initial oil stocks. The initial cost advantage for renewables implies that the economy starts with renewables during which the relative shadow price of oil decreases because capital decreases and thus at some point in time oil will be phased in alongside renewables.[26]

Fig. 6(b) extends proposition 5 for the "laissez-faire" economy to the case where, in contrast to what is stated in assumption 1, renewables have an initial cost advantages, i.e., $G(S_0) > b.$ If that is the case, the "laissez-faire" economy always starts with renewables and never phases in oil. Comparing figs. 6(a) and (b), we see immediately that for a much bigger range of initial oil stocks the social optimum is more likely to implement renewables. The reason is, of course, that the social optimum internalizes global warming externalities and the "laissez-faire" economy does not. Still, even with a carbon tax, without a breakthrough in renewables technology, it is unlikely that the economy will start with renewables.

[26] Regimes III and IV in fig. 6(a) can be interpreted in a phase-diagram setting. If we start in regime III, the stock of oil and hence the stock of CO2 does not change and capital monotonically approaches its steady state. If we start in regime IV, the stocks of oil and CO2 remain unaltered until the phase with simultaneous use starts. So the economy moves along a horizontal line. However, once it starts, the economy does not move along the dividing manifold depicted in the figure, because now also the stock of CO2 changes. For the same reason fig. 6 cannot be interpreted as a phase diagram for the other regimes. This is also why the proof of proposition 4 is not trivial.

9. Conclusion

We have analyzed optimal climate policy in a Ramsey growth model with exhaustible oil reserves, an infinitely elastic supply of renewables, stock-dependent oil extraction costs and convex climate damages and characterized the four regimes that can occur in the socially optimal outcome. The two most likely regimes occur if the economy has an initial cost advantage of oil and an initial capital stock below the carbon-free steady state. Regime I occurs if the initial stock of oil is small in which case it is optimal to have an initial oil-only phase followed by a renewables-only, carbon-free phase. A lower cost of renewables, a lower discount rate or a lower degree of intergenerational inequality aversion induce a higher long-run carbon tax which ensures that more oil is left in situ and renewables are phased in more quickly. The optimal carbon tax rises as the economy moves along its development path during the oil-only phase. The rise in the carbon tax flattens off as less accessible reserves have to be explored and the marginal cost of global warming increases as the amount of carbon in the atmosphere and the marginal utility of consumption falls. If a carbon tax is infeasible and renewables are subsidized, renewables are phased in more quickly and more oil is left in situ. However, oil is also pumped up more vigorously (a manifestation of the Green Paradox), so that the effect on global warming is ambiguous.

Regime II occurs if the initial oil stock is large in which case the social optimum has an initial oil-only phase followed with a final phase where oil and renewables are used alongside each other; oil is phased out and renewables phased in much later under “laissez faire”. The energy price starts below that in “laissez-faire”, but during the latter part of the oil-only phase rises above it. As a result, oil use is thus initially higher than “laissez faire” and is only lower in the latter part of the only-oil phase. In the final phase use of renewables is gradually ramped up. The social optimum boosts green welfare at the expense of private welfare by substantially curbing oil use and carbon emissions and thus flattening output growth during the latter part of the oil-only phase, despite the increase in oil use during the early part of this phase. Consumption under “laissez faire” is higher throughout, especially during the latter part of the oil-only phase. The social optimum also leaves more oil in situ and thus less carbon in the atmosphere.

The oil-abundant regime II also becomes relevant if the economy starts off with a high enough degree of economic development. Although the economy ends up in the same steady state, the mature economy keeps on using oil alongside renewables forever whilst the developing economy switches from using only oil to only renewables, albeit that the mature economy phases renewables in more quickly. The mature economy leaves less oil in situ and has a higher carbon tax than the developing economy for the oil-only phase, since it faces a higher social cost of carbon. In a developing economy the optimal carbon tax is gradually ramped up until the moment renewables take over from oil. In the mature economy the carbon tax first rises and then falls during the final oil-renewables phase where capital, consumption and the social cost of carbon fall. Hence, despite there being no decay of the CO2 stock, the carbon tax may decrease if the level of development is high enough.

Regime II also occurs if the initial capital stock is above its carbon-free steady state, provided that the initial oil stock is high enough and thus oil extraction costs and marginal climate damages low enough for it to remain attractive to start with oil. However, if the initial stock of oil is low enough, it is attractive to start with renewables rather than oil. If in addition the initial stock of oil is below a threshold value, regime III prevails where renewables are used forever (also if the initial capital stock is below steady state). For intermediate values of the initial oil stock with the initial capital stock still above its steady state, regime IV prevails with an initial renewables-only phase followed by a final oil-renewables phase. Until there is a breakthrough in renewables technology, these latter two regimes seem unlikely to occur.

The different regimes are the result of extraction costs being lower as oil stocks are larger, thus making oil use more attractive. However, damages also play a role: with small initial capital stocks the marginal utility of consumption is high relative to the marginal cost of climate change, which implies higher attractiveness of using oil. Regimes II and IV feature simultaneous use and overshooting.

The innovation of the paper is not only the precise characterization of which of the various regimes takes place and whether oil and renewables are used on their own or used simultaneously in the production process, but also the endogenous determination of the optimal switch time between the different phases of each regime and the optimal amount of oil to be left unused in the crust of the earth. We have also shown that the optimal carbon tax increases the amount of fossil fuel that is left in the crust of the earth and also brings forward the date that fossil fuel is phased out and carbon-free renewables are phased in compared with the "laissez-faire" outcome. We have also shown that under "laissez faire" there is never simultaneous use and capital and consumption fall towards their long-run values during the carbon-free phase despite rising during the oil-only phase. We have shown that in all possible regimes (including the market economy) there is at most one transition. We have also shown that, in spite of the absence of natural decay of the CO2 stock, the optimal carbon tax may decrease over time, due to a rapid increase of the price of capital.

We have used a stylized model to highlight the importance of endogenous determination of the time that the economy switches from fossil fuel to renewables and the optimal amount of fossil fuel to be left in situ. In practice, there may be an upward-sloping supply schedule of renewables (e.g., Sinn, 2008ab) which will introduce regimes where more and more renewables are phased in alongside oil (van der Ploeg and Withagen, 2010). There may be technical progress in renewables (e.g., Bovenberg and Smulders, 1996; Popp, 2002; Bosetti, et al., 2009; Acemoglu et al., 2012) leading to a gradual decline in the price of renewables, thus bringing forward the date of the switch from fossil fuel to renewables and kick-starting green innovation. Technical progress and population growth will affect the optimal carbon tax. Imperfect substitution between energy and other production factors is weak in the short run, but due to directed energy-saving technical change strong in the long run (Hassler, et al., 2011). Imperfect substitution between the various sources of energy also plays a role (Smulders and van der Werf, 2008; Michielsen, 2011). Natural decay of the atmospheric stock of CO2 makes the optimal carbon tax eventually fall over time (cf, Golosov et al., 2011). If coal instead of renewables is the relevant backstop, the optimal strategy is to have a more conservative oil depletion strategy and delay the time one has to switch to using coal (van der Ploeg and Withagen, 2011). The robustness of our results if global warming damages affect production multiplicatively (rather than utility additively) should be addressed with great scrutiny; more generally, the elasticity of substitution between damages and economic output (or consumption) might have an important effect on both the time profile of the optimal carbon tax. Finally, China and India are growing rapidly and have little appetite for an aggressive climate policy whilst the more mature OECD economies have more inclination to fight global warming. OPEC has monopoly power and no immediate interest in climate policy. A multi-country model will shed more light on the different tradeoffs between climate and development facing different parts of the world. We leave these issues for further research.

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Appendix 1: Proofs

Proof of propositions 3 and 4: The proofs proceed in several steps. Define total energy use $V \equiv O + R$.

(i) $F_K(K,V) - \rho - \delta > 0$ *and* $\dot{C} > 0$ *as long as* $K < K^*$.

Proof of (i): Recall that (K^*, V^*) is defined by $F_V(K^*, V^*) = b$ and $F_K(K^*, V^*) = \rho + \delta$. It follows from concavity of F that $(\rho + \delta - F_K(K,V))(K - K^*) + (b - F_V(K,V))(V - V^*) \geq 0$. We have $K < K^*$ by assumption, and $b \geq F_V(K,V)$, as required by (8b). Suppose $F_K(K,V) - \rho - \delta \leq 0$. Then $V \geq V^*$. But with $K < K^*$ and $V \geq V^*$ we have $F_K(K,V) > F_K(K^*, V^*) = \rho + \delta$, which is a contradiction. It follows from (A2a) and (A2d) that consumption is rising as well.

(ii) *Once the economy starts with only using renewables, it continues using only them (**part 1 of proposition 3**).*

Proof of (ii): Suppose that at time *0* the economy starts with using only renewables until time t_1, where oil is taken into use. At $t = 0$ we have $b \le G(S_0) + \mu(0) / \mu_K(0)$ because otherwise we would not have started with renewables only. At $t = t_1$ we have $b = G(S_0) + \mu(t_1) / \mu_K(t_1)$ in view of (8b) and (8c). . Hence $\mu(t_1) / \mu_K(t_1) \le \mu(0) / \mu_K(0)$. Note that $S(t_1) = S_0$ and $E(t_1) = E_0$. If $K(t_1) > K_0$ then $\mu(t_1) / \mu_K(t_1) > \mu(0) / \mu_K(0)$ since over time capital has become less scarce relative to oil over the interval $[0, t_1)$. This yields a contradiction. Hence $K(t_1) \le K_0$. In a renewables-only phase we have $\ddot{K} = (F_K - \delta)\dot{K} - \dot{C}$ with $\dot{C} > 0$ (see part (i) of proofs) and $F_K - \delta > 0$. Hence, once capital is decreasing along a renewables-only phase, it will keep decreasing. But this cannot go on forever because eventually the economy approaches the steady state capital stock. Hence there exists $t_2 > t_1$ with $K(t_2) = K_0$, $S(t_2) < S_0$, $E(t_2) > E_0$ and $O(t_2) > 0$. So, $\mu(t_2) / \mu_K(t_2) > \mu(0) / \mu_K(0)$ because the social cost of using oil relative to capital has increased since time zero. But then $b < G(S(t_2)) + \mu(t_2) / \mu_K(t_2)$ contradicting that $O(t_2) > 0$.

(iii) *As long as $K < K^*$ there occurs no simultaneous use.*

Proof of (iii): It follows from (8b)-(8d) that along an interval of simultaneous use we have $b = G(S) + \dfrac{\mu}{\mu_K} = G(S) + \dfrac{D'(E_0 + S_0 - S)}{(F_K(K,V) - \delta)U'(C)}$. Since *S* is decreasing, and *C* is increasing for $K < K^*$, it follows that $F_K(K,V)$ is increasing in order to maintain the equality. Since $F_V(K,V) = b$ and F is concave it follows that $\dot{K} < 0$. Suppose the simultaneous phase starts at $t_1 > 0$. Since eventually K approaches K^*, there exists $t_2 > t_1$ such that $K(t_2) = K(t_1) = K < K^*$ and $\dot{K}(t_2) > 0$. We cannot have only use of renewables at t_2 because if that would be the case we would have:

$0 < \dot{K}(t_2) = F(K, V(t_2)) - bV(t_2) - C(t_2) - \delta K <$
$F(K, V(t_1)) - bV(t_1) + (b - G(S(t_1))R(t_1) - C(t_1) - \delta K = \dot{K}(t_1) < 0$, since $V(t_1) = V(t_2)$, from $F_R(K, V(t_1)) = F_R(K, V(t_2)) = b$, and $C(t_1) < C(t_2)$ (part i). Only oil use at t_2 is excluded as well, since at t_2, with $K(t_1) = K(t_2)$, the relative marginal cost of oil is larger than at t_1, so that $b < G(S(t_2)) + \mu(t_2) / \mu_K(t_2)$ and oil is not used.

(iv) *As long as $K < K^*$ capital is increasing over time.*

Proof of (iv): In view of part (iii) we do not have to consider the case of simultaneous use. The statement is true if we are in a renewables-only phase, because once such a phase starts, it will last forever (part ii), and the economy approaches its steady state with capital monotonically increasing. So, we assume that along some interval of time with only fossil fuel use, capital is decreasing and establish a contradiction. Since the economy will eventually approach the steady state capital stock, the decrease will not be permanent. Hence, there exist instants of time $t_1 < t_2$ with $K(t_1) = K(t_2) = K < K^*$ and $S(t_1) > S(t_2)$ such that $\dot{K}(t_1) = F(K, R(t_1)) - G(S(t_1))O(t_1) - \delta K - C(t_1) < 0$ and either

Case 1: $\dot{K}(t_2) = F(K, O(t_2)) - G(S(t_2))O(t_2) - \delta K - C(t_2) > 0$ or

Case 2: $\dot{K}(t_2) = F(K, R(t_2)) - bR(t_2) - \delta K - C(t_2) > 0.$

Consumption is increasing over time (see part (i)), so that $C(t_1) < C(t_2)$. In case 1 we therefore have $F(K, O(t_1)) - G(S(t_1))O(t_1) < F(K, O(t_2)) - G(S(t_2))O(t_2)$. Moreover, $F_R(K, O(t_i)) = G(S(t_i)) + \mu(t_i)/\mu_K(t_i), i = 1, 2$. From the fact that we have the same capital stock at both instants of time but higher social cost of oil in t_2 we have $\mu(t_2)/\mu_K(t_2) > \mu(t_1)/\mu_K(t_1)$. It then follows that $F_R(K, O(t_1)) < F_R(K, O(t_2))$ so that $O(t_2) < O(t_1)$. Reduce $O(t_1)$ to $O(t_2)$. Then we have $F(K, O(t_2)) - G(S(t_1))O(t_2) = F(K, O(t_2)) - G(S(t_2))O(t_2) + (G(S(t_2)) - G(S(t_1))O(t_2)$ $> F(K, O(t_1)) - G(S(t_1))O(t_1)$. So, by decreasing $O(t_1)$ we yield more net production and less pollution. Welfare can be improved and we were not on an optimal path. In case 2 we have $F(K, R(t_2)) - bR(t_2) > F(K, O(t_1)) - G(S(t_1)O(t_1)).$ Indeed, we now have $F_R(K, R(t_2)) = b > G(S(t_1))$ and $F_O(K, O(t_1)) < b$ from the necessary conditions ($b \geq G(S(t_1)) + \mu(t_1)/\mu_K(t_1)$). This implies that $R(t_2) < O(t_1)$ and the same contradiction is reached.

(v) *There exists S_0^* such that it is optimal to start with only oil until the steady-state capital stock is reached. From there on, it is optimal to stay in the steady state*

(vi) **Proof of (v):** Fix $S_0 > 0$ and define S^* by $b = G(S^*) + \dfrac{D'(E_0 + S_0 - S^*)}{\rho U'(C^*)}$ .We assume $0 < S^* < S_0$.

Consider maximize $W = \int_0^T e^{-\rho t} \ U(C(t)) - D(E(t) \ dt + \int_T^\infty e^{-\rho t} \left[U(C^*) - D(E^*) \right] dt$ subject to:

$$\dot{K} = F(K, O) - G(S)O - C - \delta K, K(0) = K_0, K(T) = K^*,$$

$$\dot{S} = -O, S(0) = S_0, S(T) = S^*,$$

$$\dot{E} = O, E(0) = E_0, E(T) = E^* = E_0 + S_0 - S^*$$

This economy uses oil up to T and the renewables from T onwards. It is important to note that the maximization also takes place with respect to T. Assuming we have an interior solution, this gives rise to the condition $U(C(T)) - D(E(T)) = U(C^*) - D(E^*)$ implying that $C(T) = C^*$. Hence for the given initial values we maximize total welfare subject to the condition that at some instant of time the economy finds itself in a particular state and stays there for ever after. This is not the problem our economy is facing, but it is helpful as a benchmark. Suppose $K_0 = K^*$. Since $0 < S^* < S_0$ we have $b > G(S_0) + \dfrac{D'(E_0)}{\rho U'(C^*)}$. In our original economy (with renewables available as backstop), it is optimal to use only oil initially and to let capital increase (from part (iv)). This implies for the problem stated here that initially the economy will exhibit growth of capital. Then eventually the economy converges to the steady state (K^*, S^*, E^*).

Due to continuity we always have $F_O(K(T),O(T))=b$, but initially $F_O(K_0,O(0))<b$ because much oil is used. The optimal T is positive. Note that if $S^*=S_0$ the optimal T is zero.

Next, take $K_0<K^*$ and keep the initial and final oil stocks unchanged. For K_0 close enough to K^* we have overshooting and the period of time used to reach K^* for the first time is short. However, for K_0 far from K^* and given the limited available oil stock it will take long to reach K^* and overshooting will not take place. Moreover, $F_O(K^*,O(T))>b$. Applying a continuity argument we state that there exists K_0^* such that at the optimal T we have $F_O(K_0^*,O(T))=b$ and $F_O(K(t),O(t))<b$ for all $t<T$. In other words, for this specific initial capital stock it is optimal to fully extract the available oil (S_0-S^*) at some instant of time and to reach the steady state exactly at that instant of time. This also proves that for this initial capital stock, there is a unique initial oil stock that yields the same result. Hence, we could also perform the analysis not by varying the initial capital stock but by varying the initial oil stock. Hence, there exists an initial oil stock S_0^* with the property that if this is the actual initial oil stock, it is optimal to use only oil up to the moment where the carbon-free steady state is reached, but oil will not be used thereafter. This optimal transition time, given the initial capital stock, is denoted by $T^*(K_0)$ because it is fully determined by the initial capital stock.

(vi) *If* $S_0<S_0^*$, *there will be oil use initially as well but renewables take over before the carbon-free steady state is reached.*

Proof of (vi): This is evident from (v).

(vii) *If* $S_0>S_0^*$, *there will be overshooting of the carbon-free steady state. Once* $K>K^*$ *there is no phase with only use of renewables.*

Proof of (vii): First of all we show that a phase with oil use cannot be interrupted by a phase with only use of renewables. Suppose that this does not hold. Assume that at $t_1>0$ a transition takes place from oil use to use of renewables only, and that the reverse takes place at $t_2>t_1$. Then, according to (5b) and (5c) $b=G(S(t_1)+\mu(t_1)/\mu_K(t_1)=G(S(t_2))+\mu(t_2)/\mu_K(t_2)$ implying from $S(t_1)=S(t_2)$ that $\mu(t_1)/\mu_K(t_1)=\mu(t_2)/\mu_K(t_2)$. Hence the initial conditions only differ in capital. Suppose $K(t_1)>K(t_2)$. Then $\mu(t_1)/\mu_K(t_1)>\mu(t_2)/\mu_K(t_2)$ since capital is relatively scarce at t_2. Hence we obtain a contradiction. This same argument can be used to show that $K(t_1)<K(t_2)$ is ruled out. Hence $K(t_1)=K(t_2)$. Now, with equal stocks at t_1 and t_2 and starting from the same initial stocks the programs to be pursued should be equal as well, meaning that it is optimal to have renewables use only from t_1 on, a contradiction.

Second, take T to be the moment of the transition from oil use to only renewables use, so that $T>0$ and assume $S(T)>0$. Then $b=G(S(T))+\dfrac{D'(E_0+S_0-S(T))}{\rho U'(C(T))}$ and $b\le G(S(T))+\dfrac{D'(E_0+S_0-S(T))}{\rho U'(C(t))}$ for all

$t \geq T$ because we have only renewables use and because the co-state variable μ becomes a constant. We are in the case with $K(T) > K^*$. Hence, from T on the economy converges to the carbon-free steady state with consumption declining from T on. This implies that U' is increasing, which yields a contradiction. Hence, either $T = 0$ or $S(T) = 0$. Now we can show that renewables are never used. Suppose that with overshooting there exists $t_1 > 0$ such that $K(t_1) > K^*$ and $S(t_1) < S^*$. Since there is overshooting, we also have $S_0 > S_0^*$ implying $E_0 + S_0 - S(t_1) > E_0 + S_0^* - S^*$. Hence, oil has higher marginal social cost than renewables, and renewables should be taken into use before oil is exhausted, contradicting that renewables are only used after depletion.

Finally, with overshooting there renewables are never used alone. If renewables would take over at instant of time T, then it must be the case that $S(T) = 0$. Recall that S^* is left unexploited if we start from (K_0, S_0^*), where $b = G(S^*) + \dfrac{D'(E_0 + S_0^* - S^*)}{\rho U'(C^*)}$. If we start with $S_0 > S_0^*$ and we would end with $S(T) = 0$ there exists t_1 such that $S(t_1) = S^*, K(t_1) > K^*, E(t_1) > E^*$ implying that $\mu(t_1) / \mu_K(t_1) > \mu^* / \mu_K{}^*$. But then it is optimal to have renewables use already before t_1 and the oil stock is not depleted. Q.E.D.

Proof of proposition 5: It follows from (8b)-(8d) that along an interval of simultaneous use we have $b = G(S) + \dfrac{\mu}{\mu_K} = G(S) + \dfrac{D'(E_0 + S_0 - S)}{(F_K(K,V) - \delta)U'(C)}$. In the market economy $D \equiv 0$, so we have $b = G(S)$. But this implies that *S* is constant, which implies that *O* = 0, a contradiction.

If $G(S_0) < b$, it is optimal to have an initial interval with only oil use followed by an interval where renewables are used forever, so that the carbon-free economy converges to the steady state (C^*, K^*).

Proof of proposition 6: The consumer maximizes $\int_0^\infty e^{-\rho t} U(C) dt$ subject to its budget constraint $\dot{A} = rA + Y - C,$ where *A* denotes household assets, *r* the market interest rate, and *Y* wages plus profits. This yields the Euler equation $\dot{C} = \sigma C(r - \rho).$ Consider first the case where firms internalize the effect of oil depletion on extraction costs. Firms thus maximize the present value of their expected stream of profits, $\int_0^\infty \left[F \;\; K(s), O(s) + R(s) \;\; - \;\; G(S(s) + \tau(s) \;\; O(s) + Z(s) - bR(s) - I(s) \right] e^{-\int_t^s r(s')ds'} ds,$ subject to the investment accumulation equation $\dot{K} = I - \delta K$, the oil depletion equation $\dot{S} = -O$, and $O, R \geq 0,$ where *Z* stands for the carbon tax revenues that are refunded in a lump-sum manner to firms and *I* stands for aggregate investment. Defining the Hamiltonian function $\hat{H} \equiv F \;\; K, O + R \;\; - \;\; G(S) + \tau \;\; O + Z - bR$ $-I + \hat{\mu}_K \;\; I - \delta K \;\; - \hat{\mu} O,$ we obtain the first-order conditions $-1 + \hat{\mu}_K = 0,$ $F_O - G(S) - \tau - \hat{\mu} \leq 0$ and $O \geq 0,$ c.s., $F_R \leq b$ and $R \geq 0,$ c.s., $r\hat{\mu}_K - \dot{\hat{\mu}}_K = F_K - \delta \hat{\mu}_K$ and $r\hat{\mu} - \dot{\hat{\mu}} = -G'(S)O.$ This gives rise to the efficiency conditions $F_K(K, O) = r + \delta$ and $F_O \leq G(S) + \tau + \hat{\mu}$ and $O \geq 0,$ c.s., where it follows from integration that $\hat{\mu}(t) = \int_t^\infty \left[-G'\big(S(s)\big) O(s) \right] e^{-\int_t^s r(s')ds'} ds, \ \forall t \in [0,T].$ In equilibrium the government budget must be balanced, $Z = \tau O,$ and wage plus profit income equals $Y = F(K,O) - \big(G(S) + \tau\big) O - bR.$ Now turning our attention to the socially optimal outcome described by equations (1), (2), (3), (5), (6) and (7), we find $\eta(t) = \int_t^\infty \left[\frac{D'\big(E_0 + S_0 - S(s)\big)}{U'\big(C(s)\big)} - G'\big(S(s)\big) O(s) \right] e^{-\int_t^s r(s')ds'} ds, \quad \forall t \in [0,T]$. Comparing the first-order conditions of the decentralized market and the socially optimal outcome, we match them by setting $\tau = \eta - \hat{\mu}$ which gives (17). Since with time K and C rise, r and $U'(C)$ fall with time. Given that S falls with time, τ given by (17) rises with time in an economy with $K_0 < K^*$. Q.E.D.

Appendix 2: Details of solving for the optimal time paths of regimes I and II

A.1. Optimal pasting conditions, switch time and stock of oil to be left in situ

I. Initial oil and capital stocks small: the oil-only, renewables-only regime

The optimal program of regime I of proposition 4 consists of two phases.

(i) *Oil-only phase* ($0 \le t \le T$):

The initial oil-only phase is described by equations (1), (3O), (6O) and (7O). Given the switch time T, initial values for S_0 and K_0 and terminal values $C(T) = \Theta^R\big(K(T);b\big)$, where $\Theta^R(.)$ is the stable manifold of the renewables-only phase, and $p(T) = b$ at some of time T, we can solve the resulting two-point-boundary-value problem for the time paths of the initial oil-only phase. Hence, this gives $K(T)$, which is used as initial condition in the subsequent phase renewables-only phase, and $S(T)$, both as functions of the unknown switch time T.

(ii) *Renewables-only phase* ($t \ge T$):

The carbon-free economy is described by equations (1), (3R) and (7R).

(iii) *Pasting the oil-only and renewables-only phases*

To paste the two phases of the optimal program, we use three pasting conditions: (i) $p(T) = b$; (ii) K, C and S must not jump at time T; (iii) $G(S(T)) + \frac{D'(E_0 + S_0 - S(T))}{\rho U'(\Theta^R\big(K(T);b\big))} = b$ which gives

(A1) $\quad 0 < S(T) = \Upsilon(\overset{-}{b}, \overset{+}{K}(T), E_0 \overset{+}{+} S_0, \overset{-}{\rho}) < S_0.$

The optimal pasting of the two regimes is completed by solving for the optimal switch time T from the condition that (A1) must equal $S(T)$ at the end of the oil-only phase

(iv) How much oil to be left in the crust of the earth?

Fig. 2 can be used to determine the optimal amount of oil to be left in situ (14), but with $S(T)$ instead of $S(T^*)$ on the horizontal axis (and $K(T)$ endogenous). For a given $K(T)$, we see that if for ethical or prudential reasons a lower discount rate is used (cf., Stern, 2007) or the initial amounts of oil and carbon in the atmosphere are large, the *SCC* curve shifts up and thus more oil is left in situ and less carbon is emitted into the atmosphere. Furthermore, again for a given $K(T)$, a lower cost of renewables shifts down the *MCA* curve in fig. 1 and thus yields a more aggressive climate policy with a higher stock of oil left in situ. Note that $K(T)$ is endogenous as it depends on ρ, S_0, E_0 and b. Still, we see from fig. 2 that a high capital stock and thus a high rate of consumption and low marginal utility of consumption shifts the *SCC* locus upwards and thus increases the stock of oil to be left in situ.

The stock of oil left in situ at time T is endogenous as it depends on the capital stock that the carbon-free economy starts off with. However, in two cases $S(T)$ can be found directly. If utility is linear and the elasticity of intertemporal substitution is infinite, i.e., $U'(C) = \varphi$ is a constant and $\sigma \to \infty$, we have from (14) that $S(T)$ is independent of $C(T)$ or $K(T)$. Also, the "laissez-faire" outcome which does not internalize global warming damages $(D'(E) = 0)$, has a zero social cost of carbon, so that the *SCC* line is the horizontal axis of fig. 1 and the market outcome (indicated by *M*) leads to a lower stock of oil reserves left in the crust of the earth than the socially optimal outcome.[27]

II: Initial oil and capital stocks large: the oil-only, oil-renewables regime

If the initial stocks of oil and manmade capital are sufficiently high (i.e., $S_0 > S_0^*(K_0, b, \rho)$), we get regime II with the following two phases of the optimal program:

(i) *Oil-only phase* ($0 \le t \le T$):

The oil-only phase follows from (1), (3O), (6O) and (7O) and can be solved given T, the initial conditions $K(0) = K_0$ and $S(0) = S_0$ and the terminal conditions $p(T) = b$ and $C(T) = \Theta^{OR} \; K(T), b, \rho$, where $\Theta^{OR}(.)$ is the stable manifold of the oil-renewables phase defined below. This yields $K(T)$ and $S(T)$ as functions of the unknown switch time T.

(ii) *Oil- renewables phase* ($t \ge T$):

With simultaneous use of oil and renewables, we have $p = F_R(K, V) = b$ and thus $O + R = V(K, b)$. The dynamics of the stock of manmade capital and consumption are now given by (7R) and

(3OR) $\dot{K} = \tilde{F}(K, b) + \; b - G(S) \; O - \delta K - C, \quad K(T)$ given.

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[27] If we relax assumption 1 and suppose that the cost of extraction does not go to infinity as oil reserves become exhausted, oil reserves will be fully exhausted if $b > G(0) + D'(E_0 + S_0) / \rho U'(C^*)$, as then for all $C < C^*$ one has $b > G(0) + D'(E_0 + S_0) / \rho U'(C)$. At the transition time T, we have $F_R(K(T), R(T+)) = b > G(0)) + \dfrac{D'(E_0 + S_0)}{\rho U'(C(T))}$ $= F_O(K(T), O(T-))$. Hence, $R(T+) < O(T-)$ and with consumption being continuous, net investments jump downward at the transition. The jump in energy use arises from the pure state constraint $S \ge 0$ becoming binding at the transition. As a result, the optimal capital stock has a kink at the transition date.

The indifference condition between using oil and renewables follows from the optimality condition (6) given in proposition 1 and, together with $p = b$, can be used to obtain the stock of oil in situ as an increasing function of the rate of consumption, the stock of capital and the global warming challenge $(E_0 + S_0)$ and a decreasing function of the cost of renewables:

$$\text{(A2)} \qquad G(S) + \frac{D'(E_0 + S_0 - S)}{\left[\tilde{F}_K(K,b) - \delta\right]U'(C)} = b \quad \Rightarrow \quad S = S(\overset{-}{b}, \overset{+}{C}, \overset{+}{K}, E_0 \overset{+}{+} S_0).$$

The difference with (A1) is that the social rate of interest rather than the rate of time preference is used to calculate the present value of marginal global warming damages. Differentiating (A2) and using (3OR) and (7R), we obtain the oil-depletion dynamics:

$$\text{(A2}'\text{)} \qquad O = -\dot{S} = -S_C \sigma C\left[\tilde{F}_K(K,b) - \delta - \rho\right] - S_K\left[\tilde{F}(K,b) + \left(b - G(S)\right)O - \delta K - C\right].$$

Equations (3OR) and (7R) with S given by (A2) and O given by (A2′) can be solved as a two-dimensional, two-point-boundary value problem for a given switch time T and $K(T)$.

(iii) Pasting the oil-only and oil-renewables phases

The solution of the final oil-renewables phase also gives consumption at the beginning of that phase, $C(T) = \Theta^{OR}\left(K(T), b, \rho\right)$, where $\Theta^{OR}(.)$ is the stable manifold of the system. This serves as terminal condition for the oil-only phase. The switch time T is chosen such that the oil stock at the end of the initial oil-only phase matches the oil stock at the beginning of the oil-renewables phase, so using (14) we require $S(T) = S\left(b, \Theta^{OR}\left(K(T), b, \rho\right), K(T), E_0 + S_0\right)$. Similarly, we require that capital at the end of the oil-only phase must equal capital at the beginning of the oil-renewables phase.

The economy converges to the steady state of the carbon-free economy, K^*, C^* and

$S^* = \Upsilon(b, C^*, E_0 + S_0, \rho)$. Since energy prices and thus energy use has to be continuous at time T, renewables use starts with a positive amount and thus oil use must fall at time T by a corresponding amount. Oil use and renewables use are thus not continuous at time T.

A2.2. Stable manifold of the renewables-only and the renewables-oil economies

The stable manifold of the carbon-free (renewables-only) economy can be found from eliminating time and solving the resulting first-order differential equation where the steady-state values of K and C pin down the solution:

$$\frac{dC}{dK} = \frac{\sigma C\left[\tilde{F}_K(K,b) - \delta - \rho\right]}{\tilde{F}(K,b) - \delta K - C} \quad \Rightarrow \quad C = \Theta^R(K;b) \ \text{ with } \ C^* = \Theta^R(K^*;b).$$

The pasting conditions for pasting the oil-only and the carbon-free phase of regime I are given by $C(T) = \Theta^R\left(K(T);b\right)$ and $S(T)$ from (A1). Notice that for simulation of the carbon-free economy only a one-dimensional equation needs to be integrated forwards once this stable manifold is substituted into (3R), i.e., $\tilde{F}(K,b)-\delta K-C(K)$, where $K(T)$ comes from the oil-only economy.

A2.3. Solving for the boundary conditions

To connect the oil-only and the oil-renewables phase in regime I, we use $\Theta^R(.)$ to obtain the right pasting condition. The TPBVP for the oil-only phase can be solved with fourth-order Runge-Kutta integration or with a spectral decomposition algorithm for the linearized model. The Runga-Kutta algorithm is nested within a Newton-Raphson method to solve for T and $S(T)$ from $S(T)=\Upsilon(b,\Theta^R\ \ K(T);b\ \ ,E_0+S_0,\rho)$ and $S(T) = b$. To solve for regime II, we use a spectral decomposition algorithm to solve for the system (3OR) and (7O) with S given by (A2) and O given by (A2′) and nest this within a Newton-Raphson algorithm to solve for T and $S(T)$ from $S\ \ b,C(T+),K(T+),E_0+S_0\ \ =S(T-)$.

A2.4. Spectral decomposition algorithm for the linearized model

The carbon-free phase starts at time T and is given by (3R) and (7R) starting with the initial condition $K(T)$. $C(T)$ must be on the saddlepath of the carbon-free economy, which we linearize as follows:

$$C(T)=\Theta^R\ \ K(T)\ \ \cong C^*(b)+\theta\ \ K(T)-K^*(b)\ \ ,\ \Theta^R{}'=\theta\equiv\frac{1}{2}\rho+\frac{1}{2}\sqrt{(\rho+\delta)^2+4\sigma\left(\frac{1-\alpha-\beta}{1-\beta}\right)(\rho+\delta)\frac{C^*}{K^*}}>0.$$

We linearize around (S^*,K^*,C^*,b) with S^* from $b=G(S^*)+\dfrac{D'(E_0+S_0-S^*)}{\rho U'(C^*)}$. Defining $R^*=V(K^*,b)$, we get the state-space system $\dot{\underset{\sim}{x}}=\mathrm{A}\underset{\sim}{x}+\underset{\sim}{a}$, where $\underset{\sim}{x}\equiv(S-S^*,K-K^*,C-C^*,p-b)'$ and $\underset{\sim}{a}\equiv(-R^*,K^{*\alpha}R^{*\beta}-G(S^*)-\delta K^*-C^*,0,0)'$. Spectral decomposition gives $\mathrm{A}=\mathrm{M}\Lambda\mathrm{M}^{-1}=\mathrm{N}^{-1}\Lambda\mathrm{N}$ where the diagonal matrix Λ contains the eigenvalues in descending order and the matrix M contains the eigenvectors. Defining the canonical variables $\underset{\sim}{y}=\mathrm{N}\underset{\sim}{x}$ yields $\dot{\underset{\sim}{y}}=\Lambda\underset{\sim}{y}+\underset{\sim}{n},\quad \underset{\sim}{n}\equiv\mathrm{N}\underset{\sim}{a}$. The system has two eigenvalues with positive real part, collected in the vector $\underset{\sim}{\lambda}_u$, and two with negative real part, collected in $\underset{\sim}{\lambda}_s$, and thus satisfies the saddlepoint property. We thus have $\Lambda=\mathrm{diag}(\lambda_{u1}\lambda_{u2},\lambda_{s1},\lambda_{s2})$, so we get:

(A3)
$$y_{ui}(t)=e^{\lambda_{ui}(t-T)}\ \ y_{ui}(T)+\bar{n}_{ui}\ \ -\bar{n}_{ui},\quad \bar{n}_{ui}\equiv n_{ui}/\lambda_{ui},\quad i=1,2,$$
$$y_{si}(t)=e^{\lambda_{si}t}\ \ y_{si}(0)+\bar{n}_{si}\ \ -\bar{n}_{si},\quad \bar{n}_{si}\equiv n_{si}/\lambda_{si},\quad i=1,2,\quad \forall t\in[0,T].$$

Decomposing so that $\mathrm{M}=\begin{pmatrix}\mathrm{M}_{su} & \mathrm{M}_{ss}\\ \mathrm{M}_{uu} & \mathrm{M}_{us}\end{pmatrix}$ and $\underset{\sim}{x}=(\underset{\sim}{x}_s{}',\underset{\sim}{x}_u{}')'$, we write the initial conditions as follows:

(A4)
$$\underset{\sim}{x}_s(0)=\mathrm{M}_{ss}\underset{\sim}{y}_s(0)+\mathrm{M}_{su}\begin{pmatrix}e^{-\lambda_{u1}T} & 0\\ 0 & e^{-\lambda_{u2}T}\end{pmatrix}\left[\underset{\sim}{y}_u(T)+\bar{\underset{\sim}{n}}_u\right]-\bar{\underset{\sim}{n}}_u=\underset{\sim}{x}_{so}\equiv(S_0-S^*,K_0-K^*)'.$$

The terminal conditions $C(T)=\Theta\ \ K(T)$ given above and $p(T) = b$ are written as follows:

(A5)
$$\mathrm{E}\left(\mathrm{M}_{ss}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix}\left[\underset{\sim}{y}_s(0)+\underset{\sim}{\bar{n}}_s\right]-\underset{\sim}{\bar{n}}_s\right\}+\mathrm{M}_{su}\underset{\sim}{y}_u(T)\right)+\mathrm{M}_{us}\left\{\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix}\left[\underset{\sim}{y}_s(0)+\underset{\sim}{\bar{n}}_s\right]-\underset{\sim}{\bar{n}}_s\right\}$$
$$+\mathrm{M}_{uu}\underset{\sim}{y}_u(T)=\underset{\sim}{0} \text{ from } \mathrm{E}\underset{\sim}{x}_s(T)+\underset{\sim}{x}_u(T)=\underset{\sim}{0}, \quad \mathrm{E}=\begin{pmatrix} 0 & -\theta \\ 0 & 0\end{pmatrix}.$$

The initial and terminal conditions (A4) and (A5) can be solved as follows:

$$\begin{pmatrix}\underset{\sim}{y}_s(0) \\ \underset{\sim}{y}_u(T)\end{pmatrix}=\begin{pmatrix} \mathrm{M}_{ss} & \mathrm{M}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T}\end{pmatrix} \\ (\mathrm{M}_{us}+\mathrm{EM}_{ss})\begin{pmatrix} e^{\lambda_{s1}T} & 0 \\ 0 & e^{\lambda_{s2}T}\end{pmatrix} & \mathrm{M}_{uu}+\mathrm{EM}_{su}\begin{pmatrix} e^{-\lambda_{u1}T} & 0 \\ 0 & e^{-\lambda_{u2}T}\end{pmatrix}\end{pmatrix}^{-1}\begin{pmatrix} \underset{\sim}{x}_{s0}+\mathrm{M}_{su}\begin{pmatrix} 1-e^{-\lambda_{u1}T} & 0 \\ 0 & 1-e^{-\lambda_{u2}T}\end{pmatrix}\underset{\sim}{n}_u \\ \mathrm{M}_{us}+\mathrm{EM}_{ss}\begin{pmatrix} 1-e^{\lambda_{s1}T} & 0 \\ 0 & 1-e^{\lambda_{s2}T}\end{pmatrix}\underset{\sim}{n}_s\end{pmatrix}.$$

Given this we can calculate $\underset{\sim}{y}$ from (A3) and thus finally obtain $\underset{\sim}{x}=\mathrm{M}\underset{\sim}{y}$ for $\forall t\in[0,T]$. The resulting solution trajectories satisfy the necessary initial and terminal conditions, (A4) and (A5). To obtain the switch time, we use (A1) and solve for time *T* from $b=G(x_{s1}(T)+S^*)+\dfrac{D'(E_0+S_0-x_{s1}(T)-S^*)}{\rho U'(x_{u1}(T)+C^*)}$. This procedure implies that *S*(*T*) depends on *C*(*T*). Given *K*(*T*) obtained from the fossil-fuel economy, the carbon-free economy can be found from multiple shooting or directly from linearization:

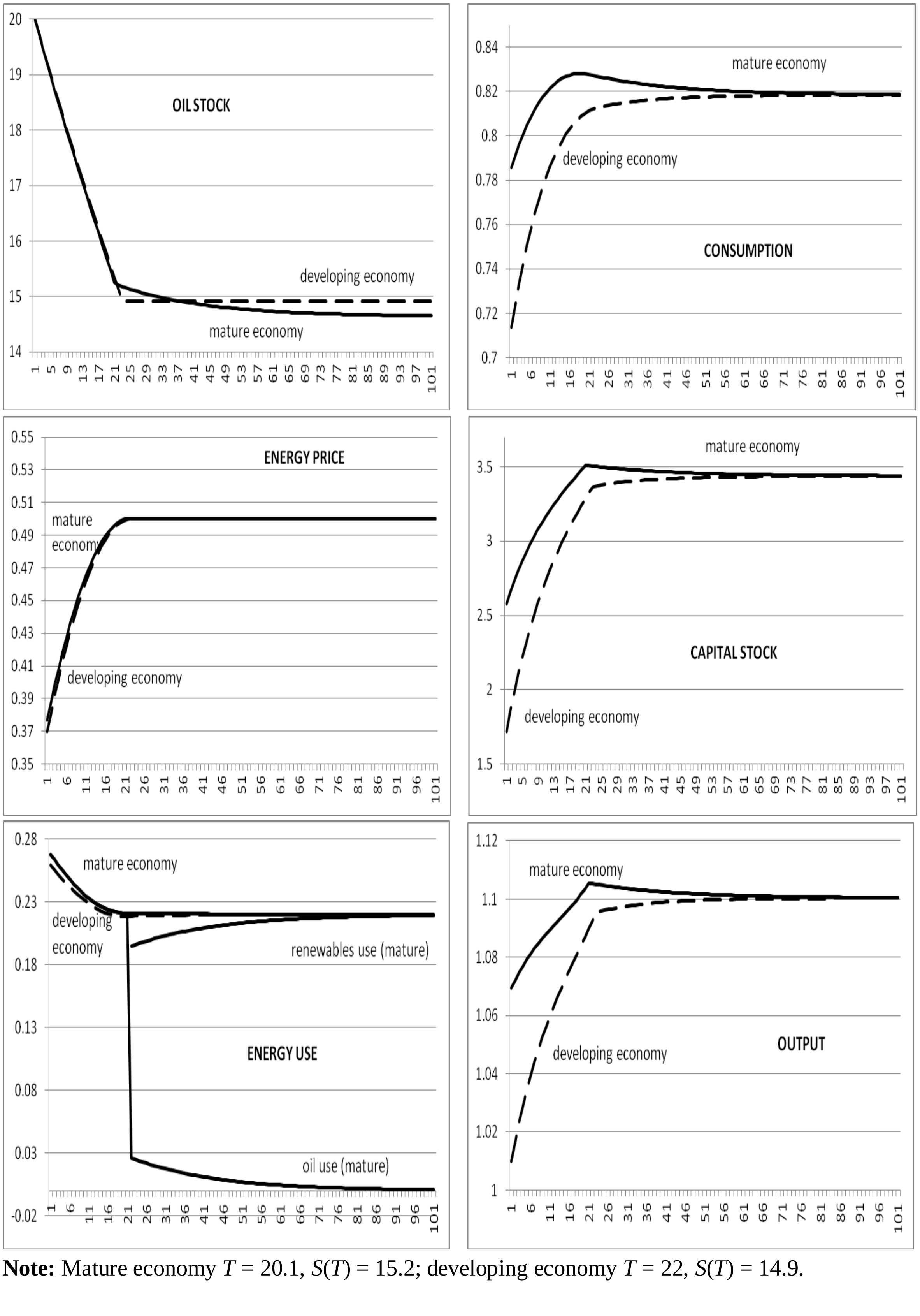
$$\dot{K}(t)=K^*+(\rho-\theta)\left[K(t)-K^*\right] \quad \Rightarrow \quad K(t)=K^*+e^{(\rho-\theta)(t-T)}\left[K(T)-K^*\right] \text{ and}$$
$$C(t)=C^*+\theta e^{(\rho-\theta)(t-T)}\left[K(T)-K^*\right], \quad \forall t\geq T, \text{ where } \theta>\rho.$$

We have also tried a fourth-order Runge-Kutta algorithm to solve (1), (3O), (6O) and (7O) given *K*(0) = K_0, *S*(0) = S_0 and guesses for *C*(0) and *p*(0); and Gauss-Newton iterations to adjust *C*(0), *p*(0) and *T* to satisfy *p*(*T*) = *b*, $b-G(S_T)=\dfrac{D'(E_0+S_0-S_T)}{\rho U'\ C(N)}$, and $C(T)=\Theta\ K(T)$. This was numerically sensitive, hence we report the results from our linearized model.

Appendix 3: Overshooting in a mature economy

If we start off with S_0 = 20 and $K_0=0.75K^*$, we get $S_0^*=18.7<20.8$ and $T^*=16.4<24.3$. The paths for the social optimum starting with K_0 = 0.5 *K** and starting with K_0 = 0.75 *K** are shown in fig. A1. They are qualitatively very similar to the paths of fig. 4 for an oil-abundant economy. A mature economy starts with a higher rate of consumption, but ends up with the same capital stock and rate of consumption in steady state. The rate of consumption overshoots first and sometime later capital overshoots its steady-state value; this does not occur in a developing, oil-scarce economy. In the mature economy $T=20.1>T^*=16.4$, but in the developing economy $T=22>T^*=24.3$.

Figure A1: Comparing social optimum for a mature economy with a developing economy



Note: Mature economy T = 20.1, $S(T)$ = 15.2; developing economy T = 22, $S(T)$ = 14.9.

The mature economy leaves less oil in the crust of the earth than the developing economy, which results from oil use throughout the only-oil phase being less than in the developing economy and no oil being use alongside renewables in the final phase. The more negative impact on the climate in a developing economy is less as, despite advancing more rapidly along its development path, oil use is somewhat less in the oil-only phase and zero in the final renewables-only phase. This is why the mature economy has higher carbon tax than the developing economy for the oil-only phase.

Appendix 4: General characterization of all regimes

We consider three cases. We start with $K_0 = K^*$. Let us assume that the equation $G(S) + \frac{D'(E_0)}{\rho U'(C^*)} = b$ has a solution, which we denote by $S_0(K^*)$. We might consider the case of no solution as well, but leave this to the reader. If the actual initial resource stock equals $S_0(K^*)$ then it is optimal to remain in the carbon free steady state and not to use oil ever. Indeed, in this regime, the Hotelling rent μ_S is zero and the marginal cost of carbon μ_E is constant. So, $\mu = D'(E_0)/\rho$ is the total discounted marginal damage of pollution. All the necessary conditions are satisfied. And the proposed programme is optimal. We define $\mu_K(0; S_0(K^*))$ and $\mu(0; S_0(K^*))$ as the (initial) shadow prices corresponding with $(K^*, S_0(K^*))$. Note that $\frac{\mu(0; S_0(K^*))}{\mu_K(0; S_0(K^*))} = \frac{D'(E_0)}{\rho U'(C^*)}$.

Next, we consider the case of a fixed $K_0 > K^*$. Suppose the initial conditions are such that $b < G(S_0) + \frac{D'(E_0)}{\rho U'(C^*)}$. Hence $S_0 < S_0(K^*)$ .It is optimal to use only renewables throughout. This program satisfies all the necessary conditions. Take $\mu = D'(E_0)/\rho$, $\mu_K = U'(C)$ and note that, starting from $K_0 > K^*$, capital and consumption are monotonically decreasing toward the steady state, so that $\mu_K < U'(C^*)$, and the oil stock remains constant.

If the initial conditions are such that $b > G(S_0) + \frac{D'(E_0)}{\rho U'(C^*)}$, then the initial resource stock is relatively abundant ($S_0 > S_0(K^*)$). Keep $S_0 > S_0(K^*)$ fixed and vary the initial capital stock. By increasing the initial capital stock its shadow price relative to the shadow price of oil can be made arbitrarily small, implying that for K_0 large enough $b < G(S_0) + \mu(0)/\mu_K(0)$ so that it is optimal to start with renewables. Along this initial phase, the oil stock and the CO2 stock remain at their initial values and capital decreases. It cannot be optimal to use renewables throughout because then the economy ends up in the steady state, where $b > G(S_0) + \mu(\infty)/\mu_K(\infty)$, because $S_0 > S_0(K^*)$ and therefore $\mu(\infty)/\mu_K(\infty) < \mu(0; S_0(K^*))/\mu_K(0; S_0(K^*))$ .Hence at some point in time T we have $b = G(S_0) + \mu(T)/\mu_K(T)$. From that instant of time oil is phased in alongside renewables. The economy converges to the carbon free steady state, where some oil is left unexploited. Hence, for the given $S_0 > S_0(K^*)$ there exists a threshold level for capital. For larger S_0 we can perform the same exercise and will find a higher threshold level for capital, because at the threshold we should have $b = G(S_0) + \mu(0) / \mu_K(0).$ Hence, with a higher initial oil stock (and therefore smaller marginal extraction cost) should correspond a relatively high capital stock, in order to have a small $\mu(0) / \mu_K(0)$ So we find an increasing locus of points such that if, by coincidence, we start on this curve, it is optimal to have simultaneous use forever. If the initial conditions are such that we start above the curve then it is optimal to start with using only oil and using renewables and oil thereafter. This is so because we will then never enter the region where only renewables are used. We have to cross the dividing curve, that in an oil only phase is moving upward, because the sock of pollutants increases. This fully characterizes the economy that start with a high capital stock.

Let us now take a fixed $K_0 < K^*$. In the appendix we prove that there cannot be simultaneous use as long as capital is smaller than K^*. The idea is simply that capital will grow. With simultaneous use the oil stock will decrease and the stock of pollutants will increase. Hence, the price of oil relative to capital will increase. But this is incompatible with the requirement that $b = G(S) + \mu / \mu_K$.

Suppose $S_0 = S_0(K^*)$. Capital is scarcer than in the previous case, leading to a larger shadow price relative to the shadow price of oil. Hence, we have

$$G(S_0(K^*)) + \mu(0) / \mu_K 0) < b = G(S_0(K^*)) + \mu(0; S_0(K^*)) / \mu_K(0; S_0(K^*)).$$

We are, therefore, in a region where it is optimal to start with only oil. We may then determine the initial oil stock where the economy is indifferent between starting with only oil or only renewables. This oil stock is the solution of $b = G(S_0(K_0)) + \mu(0; S_0(K_0)) / \mu_K(0; S_0(K_0))$. We have $S_0(K_0) < S_0(K^*)$ because otherwise extraction cost would be smaller than at $S_0(K^*)$ and the relative shadow price would be smaller as well. Hence, there is an upward sloping curve which is a dividing curve between starting with only oil and starting with only renewables. . Fig. 6 displays another dividing curve between the regime of oil to renewables use without overshooting and the regime with overshooting, as discussed in section 4. This curve slopes downwards. Why this is so can be seen as follows. A necessary condition for overshooting is that oil is cheaper than the backstop at any point in time before K^* is reached. Given $K_0 < K^*$ and if $S_0 \leq S_0(K^*)$ one can arrive in K^* at some T only with $S(T) < S_0(K^*)$ and $E(T) > E_0$ meaning that $\mu(T) / \mu_K(T) > \mu(0; S_0(K^*)) / \mu_K(0; S_0(K^*))$. Hence, $b < G(S(T)) + \mu(T) / \mu_K(T)$ so that it would have been profitable to switch to renewables earlier.